1.4 Constant Propagation

Idea:

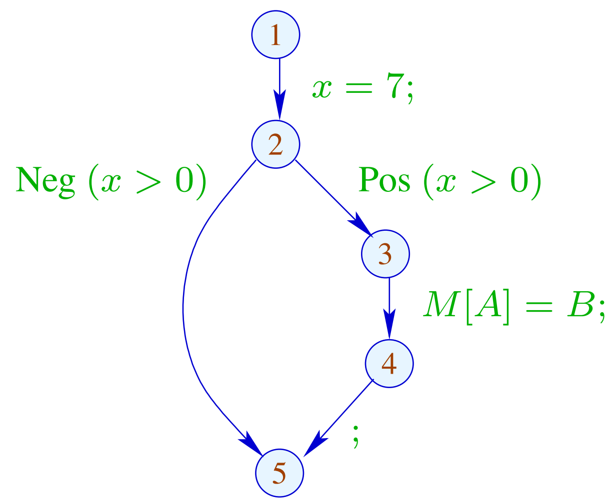
Execute as much of the code at compile-time as possible!

Example:

$$x = 7;$$

$$\text{if } (x > 0)$$

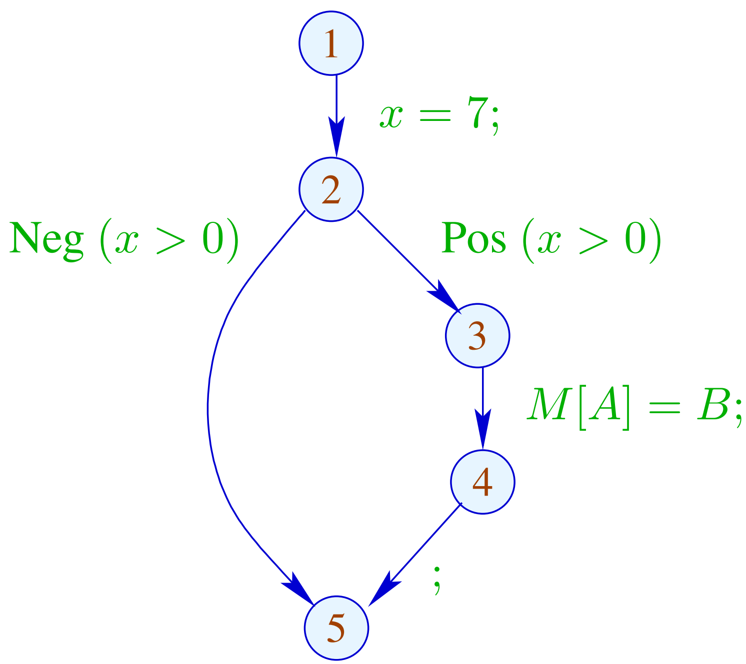
$$\qquad M[A] = B;$$



Obviously, x has always the value 7

Thus, the memory access is always executed

Goal:



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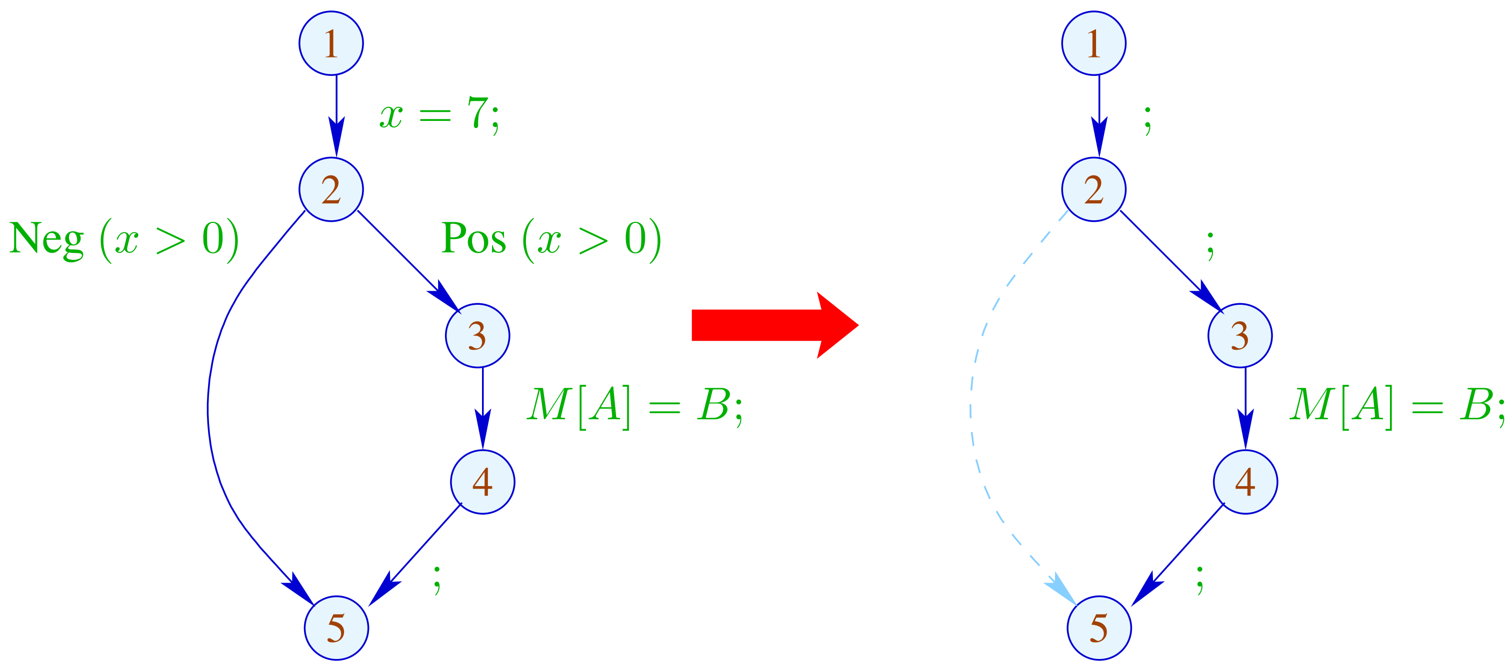

Idea:

Design an analysis that for everyprogram point u, determines the values that variables definitely have;

As a sideeffect, it also tells whether u can be reached at all

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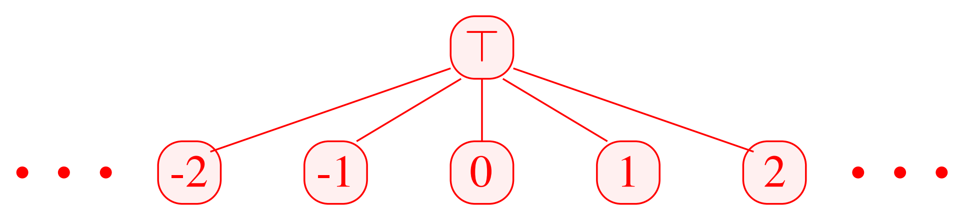
Design an analysis that for every program u, determines the values that variables definitely have;

As a sideeffect, it also tells whether u can be reached at all

The complete lattice is constructed in two steps.

(1) The potential values of variables:

$$\mathbb{Z}^{\top} = \mathbb{Z} \cup \{\top\} \qquad \text{with} \quad x \sqsubseteq y \quad \text{iff } y = \top \text{ or } x = y$$



Caveat: $\mathbb{Z}^{\top}$ is not a complete lattice in itself

(2) $\mathbb{D} = (\mathit{Vars} \to \mathbb{Z}^{\top})_{\perp} = (\mathit{Vars} \to \mathbb{Z}^{\top}) \cup \{\perp\}$

// $\perp$ denotes: "not reachable"

with $D_1 \sqsubseteq D_2$ iff $\perp = D_1$ or

$D_1\, x \sqsubseteq D_2\, x \quad (x \in \mathit{Vars})$

Remark: $\mathbb{D}$ is a complete lattice

Caveat: $\mathbb{Z}^\top$ is not a complete lattice in itself

(2) $\mathbb{D} = (\mathit{Vars} \to \mathbb{Z}^\top)_\bot = (\mathit{Vars} \to \mathbb{Z}^\top) \cup \{\bot\}$

// $\bot$ denotes: "not reachable"

with $D_1 \sqsubseteq D_2$ iff $\bot = D_1$ or

$D_1\, x \sqsubseteq D_2\, x \quad (x \in \mathit{Vars})$

Remark: $\mathbb{D}$ is a complete lattice

Consider $X \subseteq \mathbb{D}$. W.l.o.g., $\bot \not\in X$.

Then $X \subseteq \mathit{Vars} \to \mathbb{Z}^\top$.

If $X = \emptyset$, then $\bigsqcup X = \bot \in \mathbb{D}$

If $X \neq \emptyset$, then $\bigsqcup X = D$ with

$$
\begin{aligned}
D\,x &= \bigsqcup\{f\,x \mid f \in X\} \\
&= \begin{cases} z & \text{if} \quad f\,x = z \quad (f \in X) \\ \top & \text{otherwise} \end{cases}
\end{aligned}
$$

:-))

If $X \neq \emptyset$, then $\bigsqcup X = D$ with

$$D\,x = \bigsqcup\{f\,x \mid f \in X\} = \begin{cases} z & \text{if} \quad f\,x = z \quad (f \in X) \\ \top & \text{otherwise} \end{cases}$$

:-))

For every edge $k = (\_, lab, \_)$, construct an effect function $[\![k]\!]^\sharp = [\![lab]\!]^\sharp : \mathbb{D} \to \mathbb{D}$ which simulates the concrete computation.

Obviously, $[\![lab]\!]^\sharp \perp = \perp$ for all lab

Now let $\perp \neq D \in \mathit{Vars} \to \mathbb{Z}^\top$.

Idea:

- We use D to determine the values of expressions.

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  $\Longrightarrow$

  We must replace the concrete operators $\Box$ by abstract operators $\Box^\sharp$ which can handle $\top$:

$$a \,\Box^\sharp\, b = \begin{cases} \top & \text{if} \quad a = \top \text{ or } b = \top \\ a \,\Box\, b & \text{otherwise} \end{cases}$$

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- The abstract operators allow to define an abstract evaluation of expressions:

$$[\![e]\!]^\sharp \;:\; (Vars \to \mathbb{Z}^\top) \to \mathbb{Z}^\top$$

Abstract evaluation of expressions is like the concrete evaluation — but with abstract values and operators. Here:

$$
\begin{array}{lll}
[\![c]\!]^\sharp\, D & = & c \\
[\![e_1 \,\Box\, e_2]\!]^\sharp\, D & = & [\![e_1]\!]^\sharp\, D \,\Box^\sharp\, [\![e_2]\!]^\sharp\, D
\end{array}
$$

... analogously for unary operators

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... analogously for unary operators

Example: $D = \{x \mapsto 2, y \mapsto \top\}$

$$\begin{array}{lcl} [\![x + 7]\!]^\sharp\, D & = & [\![x]\!]^\sharp\, D \;+^\sharp\; [\![7]\!]^\sharp\, D \\ & = & 2 \;+^\sharp\; 7 \\ & = & 9 \\ [\![x - y]\!]^\sharp\, D & = & 2 \;-^\sharp\; \top \\ & = & \top \end{array}$$

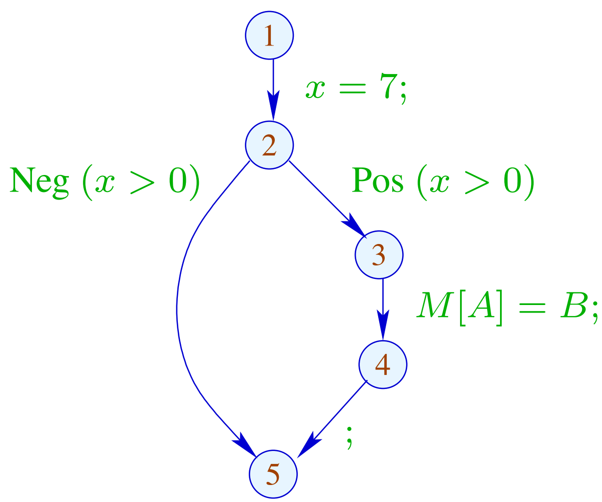
Thus, we obtain the following effects of edges $[\![lab]\!]^\sharp$:

$$
\begin{array}{lll}
[\![;]\!]^\sharp\, D & = & D \\
[\![\mathrm{Pos}\,(e)]\!]^\sharp\, D & = & \left\{\begin{array}{ll} \bot & \text{if} \quad 0 = [\![e]\!]^\sharp\, D \\ D & \text{otherwise} \end{array}\right. \\
[\![\mathrm{Neg}\,(e)]\!]^\sharp\, D & = & \left\{\begin{array}{ll} D & \text{if} \quad 0 \sqsubseteq [\![e]\!]^\sharp\, D \\ \bot & \text{otherwise} \end{array}\right. \\
[\![x = e;]\!]^\sharp\, D & = & D \oplus \{x \mapsto [\![e]\!]^\sharp\, D\} \\
[\![x = M[e];]\!]^\sharp\, D & = & D \oplus \{x \mapsto \top\} \\
[\![M[e_1] = e_2;]\!]^\sharp\, D & = & D
\end{array}
$$

... whenever $D \neq \bot$

At *start*, we have $D_\top = \{x \mapsto \top \mid x \in Vars\}$.

Example:



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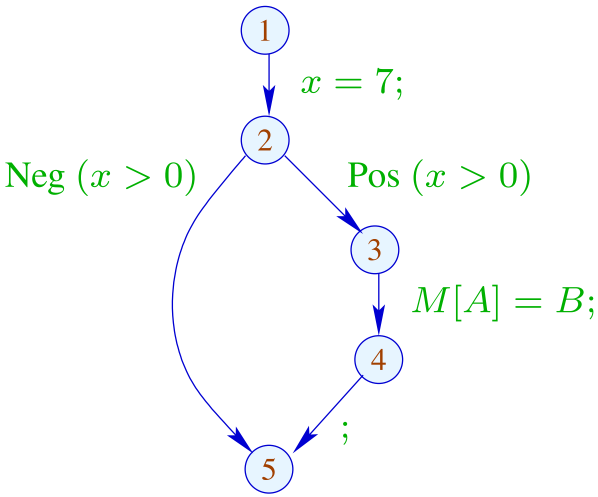

1	$\{x \mapsto \top\}$
2	$\{x \mapsto 7\}$
3	$\{x \mapsto 7\}$
4	$\{x \mapsto 7\}$
5	$\bot \sqcup \{x \mapsto 7\} = \{x \mapsto 7\}$

The abstract effects of edges $[\![k]\!]^\sharp$ are again composed to the effects of paths $\pi = k_1 \dots k_r$ by:

$$[\![\pi]\!]^\sharp = [\![k_r]\!]^\sharp \circ \dots \circ [\![k_1]\!]^\sharp \quad : \mathbb{D} \to \mathbb{D}$$

Idea for Correctness: Abstract Interpretation

Cousot, Cousot 1977

Establish a description relation Δ between the concrete values and their descriptions with:

$$x \,\Delta\, a_1 \quad \wedge \quad a_1 \sqsubseteq a_2 \quad \Longrightarrow \quad x \,\Delta\, a_2$$

Concretization: $\gamma\, a = \{x \mid x \,\Delta\, a\}$

// returns the set of described values

(1) Values: $\Delta \subseteq \mathbb{Z} \times \mathbb{Z}^\top$

$$z \,\Delta\, a \quad \text{iff} \quad z = a \vee a = \top$$

Concretization:

$$\gamma\, a = \begin{cases} \{a\} & \text{if} \quad a \sqsubset \top \\ \mathbb{Z} & \text{if} \quad a = \top \end{cases}$$

(1) Values: $\Delta \subseteq \mathbb{Z} \times \mathbb{Z}^{\top}$

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(2) Variable Bindings: $\Delta \subseteq (Vars \to \mathbb{Z}) \times (Vars \to \mathbb{Z}^{\top})_{\perp}$

$$\rho \,\Delta\, D \quad \text{iff} \quad D \neq \perp \wedge \rho\, x \sqsubseteq D\, x \quad (x \in Vars)$$

Concretization:

$$\gamma\, D = \begin{cases} \emptyset & \text{if} \quad D = \perp \\ \{\rho \mid \forall\, x : \; (\rho\, x) \,\Delta\, (D\, x)\} & \text{otherwise} \end{cases}$$

Example: $\{x \mapsto 1, y \mapsto -7\} \;\Delta\; \{x \mapsto \top, y \mapsto -7\}$

(3) States:

$$\Delta \;\subseteq\; ((\mathit{Vars} \to \mathbb{Z}) \times (\mathbb{N} \to \mathbb{Z})) \times (\mathit{Vars} \to \mathbb{Z}^{\top})_{\perp}$$

$$(\rho, \mu) \,\Delta\, D \qquad \text{iff} \qquad \rho \,\Delta\, D$$

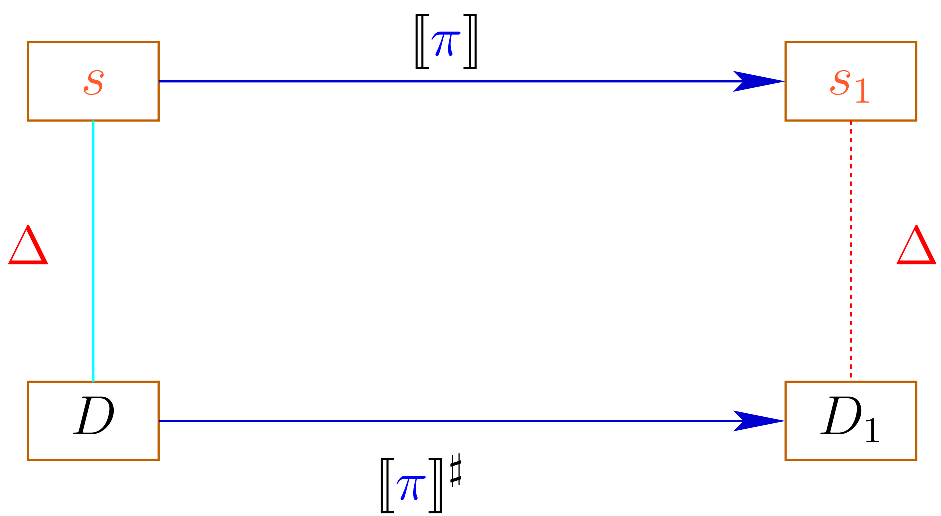
Concretization:

$$\gamma\, D = \begin{cases} \emptyset & \text{if} \quad D = \perp \\ \{(\rho, \mu) \mid \forall\, x :\; (\rho\, x) \;\Delta\; (D\, x)\} & \text{otherwise} \end{cases}$$

We show local correctness:

$(*)$ If $s \,\Delta\, D$ and $[\![\pi]\!]\, s$ is defined, then:

$$([\![\pi]\!]\, s) \;\; \Delta \;\; ([\![\pi]\!]^\sharp\, D)$$



The abstract semantics simulates the concrete semantics

In particular:

$$[\![\pi]\!]\; s \in \gamma\left([\![\pi]\!]^\sharp\, D\right)$$

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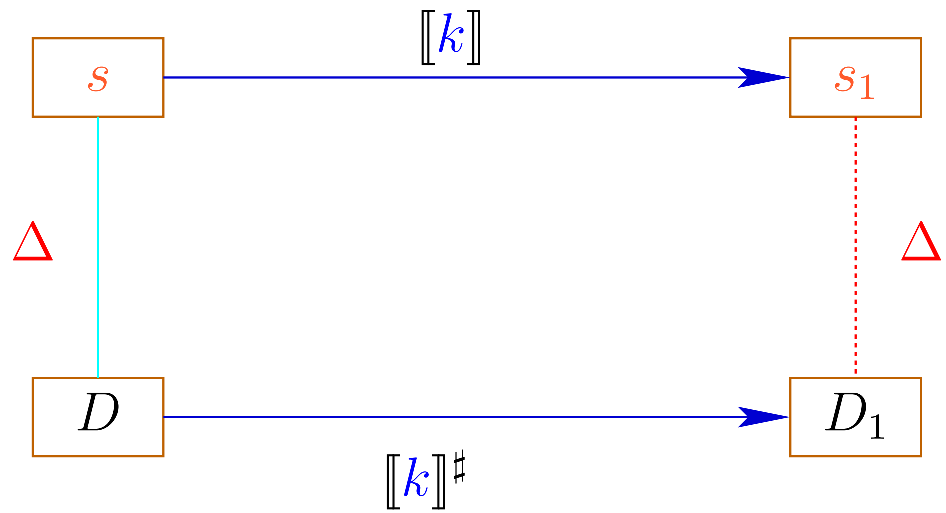
$$[\![\pi]\!]\, s \in \gamma\,([\![\pi]\!]^\sharp\, D)$$

In practice, this means for example that $D\,x = -7$ implies:

$$\begin{array}{rrcll} & \rho'\,x & = & -7 & \text{for all} \quad \rho' \in \gamma\,D \\ \Longrightarrow & \rho_1\,x & = & -7 & \text{for} \quad (\rho_1, \_) = [\![\pi]\!]\, s \end{array}$$

To prove $(*)$, we show for every edge k :

$(**)$



Then $(*)$ follows by induction

To prove $(**)$, we show for every expression e :

$(* * *)$ $\quad ([\![e]\!]\, \rho) \;\; \Delta \;\; ([\![e]\!]^\sharp\, D)$ $\quad$ whenever $\quad \rho \, \Delta \, D$

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To prove $(* * *)$, we show for every operator $\Box$:

$$(x \,\Box\, y) \;\Delta\; (x^\sharp \,\Box^\sharp\, y^\sharp) \qquad \text{whenever} \quad x \,\Delta\, x^\sharp \wedge y \,\Delta\, y^\sharp$$

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This precisely was how we have defined the operators $\Box^\sharp$

Now, $(**)$ is proved by case distinction on the edge labels *lab* .

Let $s = (\rho, \mu) \;\Delta\; D$. In particular, $\bot \neq D \;:\; \mathit{Vars} \to \mathbb{Z}^\top$

Case $\boxed{x = e;}$:

$$\rho_1 = \rho \oplus \{x \mapsto [\![e]\!]\,\rho\} \qquad \mu_1 = \mu$$

$$D_1 = D \oplus \{x \mapsto [\![e]\!]^\sharp\, D\}$$

$$\Longrightarrow \qquad (\rho_1, \mu_1) \;\Delta\; D_1$$

Case $\boxed{x = M[e];}$:

$$\rho_1 = \rho \oplus \{x \mapsto \mu\,([\![e]\!]^\sharp \rho)\} \qquad \mu_1 = \mu$$

$$D_1 = D \oplus \{x \mapsto \top\}$$

$$\Longrightarrow \qquad (\rho_1, \mu_1)\ \Delta\ D_1$$

Case $\boxed{M[e_1] = e_2;}$:

$$\rho_1 = \rho \qquad \mu_1 = \mu \oplus \{[\![e_1]\!]^\sharp \rho \mapsto [\![e_2]\!]^\sharp \rho\}$$

$$D_1 = D$$

$$\Longrightarrow \qquad (\rho_1, \mu_1)\ \Delta\ D_1$$

Case $\boxed{\text{Neg}(e)}$: $(\rho_1, \mu_1) = s$ where:

$$
\begin{array}{rccl}
 & 0 & = & [\![e]\!]\, \rho \\
 & & \Delta & [\![e]\!]^\sharp\, D \\
\Longrightarrow & 0 & \sqsubseteq & [\![e]\!]^\sharp\, D \\
\Longrightarrow & \bot & \neq & D_1 \ = \ D \\
\Longrightarrow & \multicolumn{3}{l}{(\rho_1, \mu_1)\ \Delta\ D_1}
\end{array}
$$

Case $\boxed{\text{Pos}(e)}$: $(\rho_1, \mu_1) = s$ where:

$$
\begin{array}{rccc}
& 0 & \neq & [\![e]\!]\,\rho \\
& & \Delta & [\![e]\!]^\sharp\,D \\
\Longrightarrow & 0 & \neq & [\![e]\!]^\sharp\,D \\
\Longrightarrow & \bot & \neq & D_1 \;=\; D \\
\Longrightarrow & \multicolumn{3}{l}{(\rho_1, \mu_1)\;\Delta\;D_1}
\end{array}
$$

:-)

We conclude: The assertion $(*)$ is true

The MOP-Solution:

$$\mathcal{D}^*[v] \;=\; \bigsqcup\{[\![\pi]\!]^\sharp\, D_\top \mid \pi : start \to^* v\}$$

where $D_\top\, x = \top \qquad (x \in Vars)$.

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By $(*)$, we have for all initial states s and all program executions π which reach v :

$$([\![\pi]\!]\; s) \;\;\Delta\;\; (\mathcal{D}^*[v])$$

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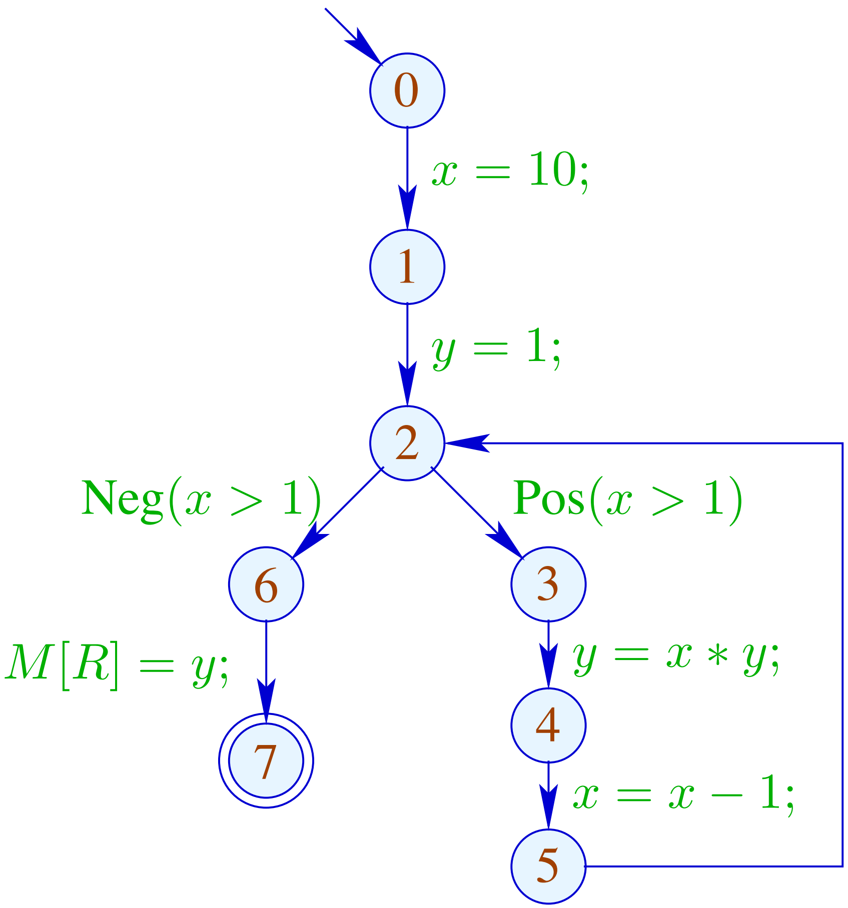
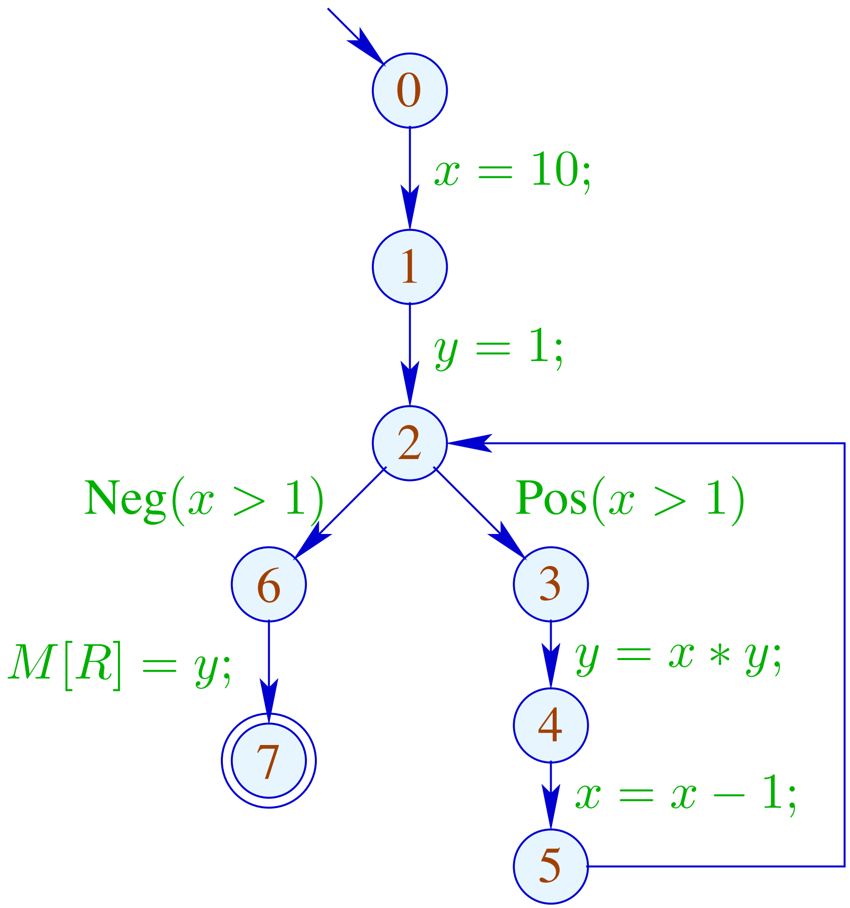
In order to approximate the MOP, we use our constraint system

Example:

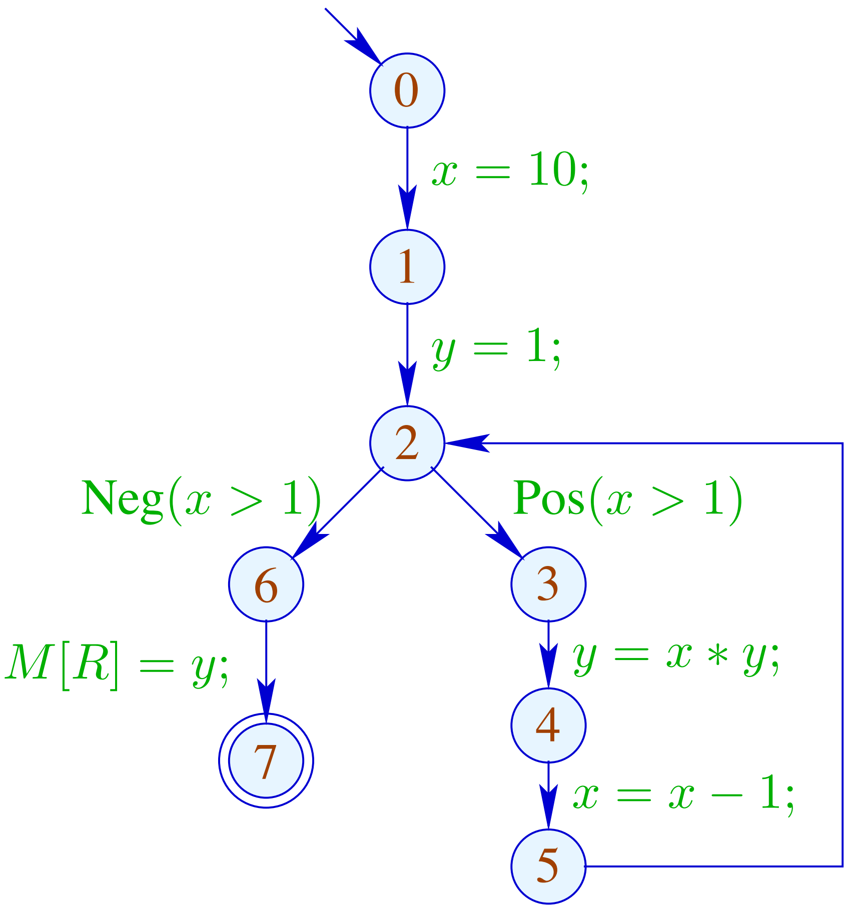


Example:



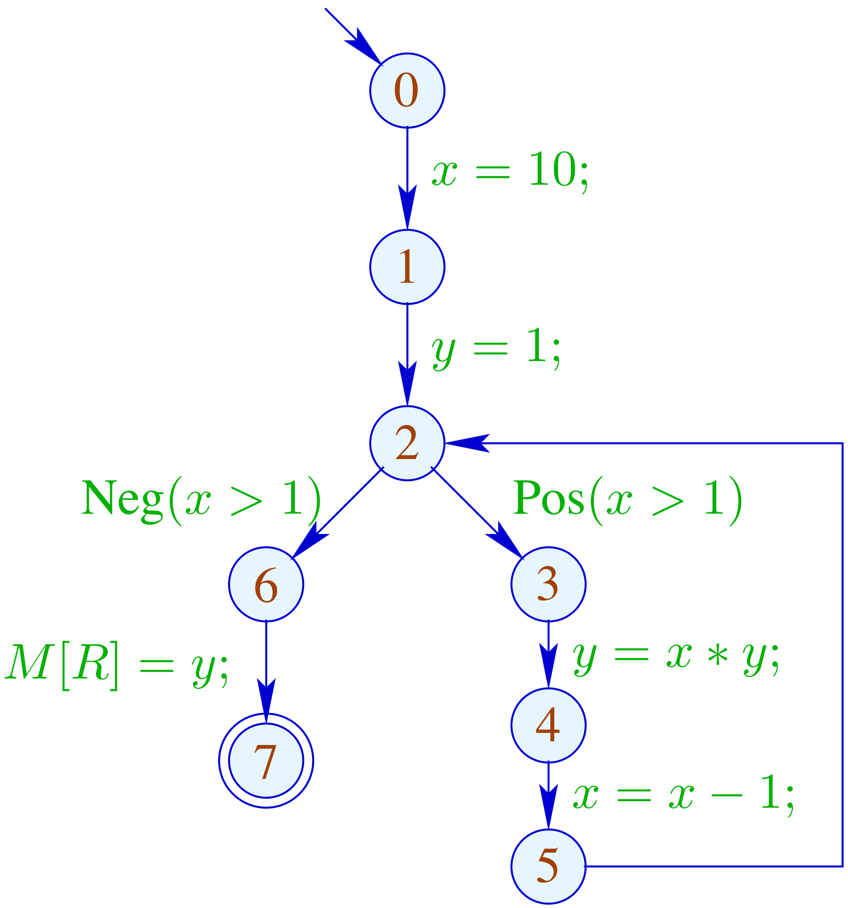
	1	
	x	y
0	$\top$	$\top$
1	10	$\top$
2	10	1
3	10	1
4	10	10
5	9	10
6	$\bot$	
7	$\bot$	

Example:



	1		2	
	x	y	x	y
0	$\top$	$\top$	$\top$	$\top$
1	10	$\top$	10	$\top$
2	10	1	$\top$	$\top$
3	10	1	$\top$	$\top$
4	10	10	$\top$	$\top$
5	9	10	$\top$	$\top$
6	$\bot$		$\top$	$\top$
7	$\bot$		$\top$	$\top$

Example:



	1		2		3	
	x	y	x	y	x	y
0	$\top$	$\top$	$\top$	$\top$	dito	
1	10	$\top$	10	$\top$		
2	10	1	$\top$	$\top$		
3	10	1	$\top$	$\top$		
4	10	10	$\top$	$\top$		
5	9	10	$\top$	$\top$		
6	$\bot$		$\top$	$\top$		
7	$\bot$		$\top$	$\top$		

Conclusion:

Although we compute with concrete values, we fail to compute everything.

The fixpoint iteration is guaranteed to terminate:

For n program points and m variables, we maximally need: $n \cdot (m + 1)$ rounds

Caveat:

The effects of edge are not distributive !!!

Counter Example: $f = [\![x = x + y;]\!]^\sharp$

$$
\begin{array}{llcl}
\textbf{Let} & D_1 & = & \{x \mapsto 2, y \mapsto 3\} \\
 & D_2 & = & \{x \mapsto 3, y \mapsto 2\} \\
\textbf{Dann} & f\,D_1 \sqcup f\,D_2 & = & \{x \mapsto 5, y \mapsto 3\} \sqcup \{x \mapsto 5, y \mapsto 2\} \\
 & & = & \{x \mapsto 5, y \mapsto \top\} \\
 & & \neq & \{x \mapsto \top, y \mapsto \top\} \\
 & & = & f\,\{x \mapsto \top, y \mapsto \top\} \\
 & & = & f\,(D_1 \sqcup D_2)
\end{array}
$$

:-((

We conclude:

The least solution $\mathcal{D}$ of the constraint system in general yields only an upper approximation of the MOP, i.e.,

$$\mathcal{D}^*[v] \quad \sqsubseteq \quad \mathcal{D}[v]$$

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The least solution $\mathcal{D}$ of the constraint system in general yields only an upper approximation of the MOP, i.e.,

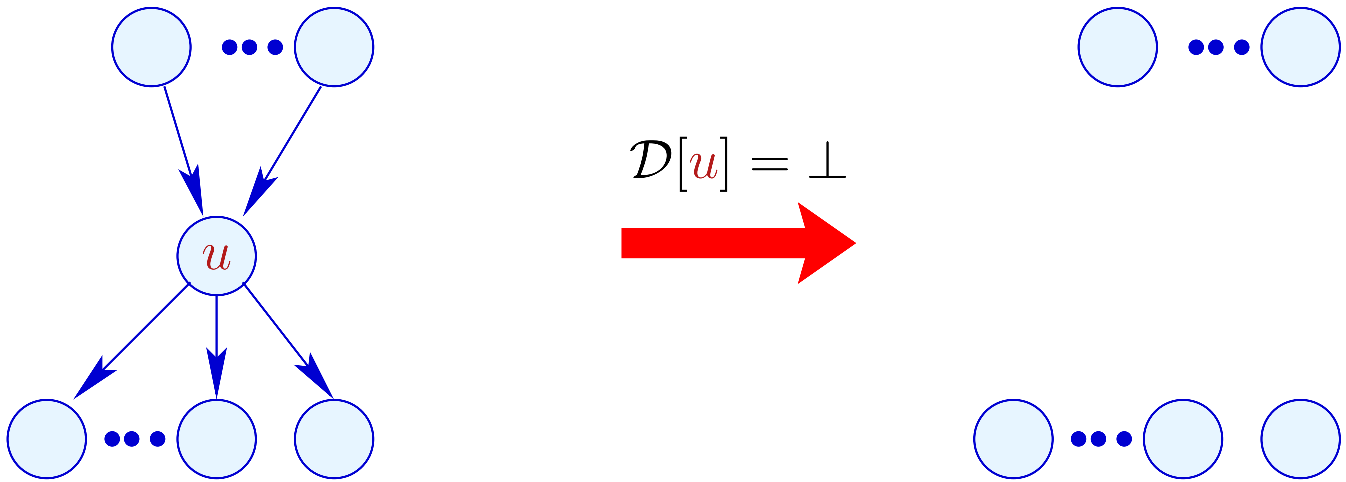
$$\mathcal{D}^*[v] \sqsubseteq \mathcal{D}[v]$$

As an upper approximation, $\mathcal{D}[v]$ nonetheless describes the result of every program execution π that reaches v :

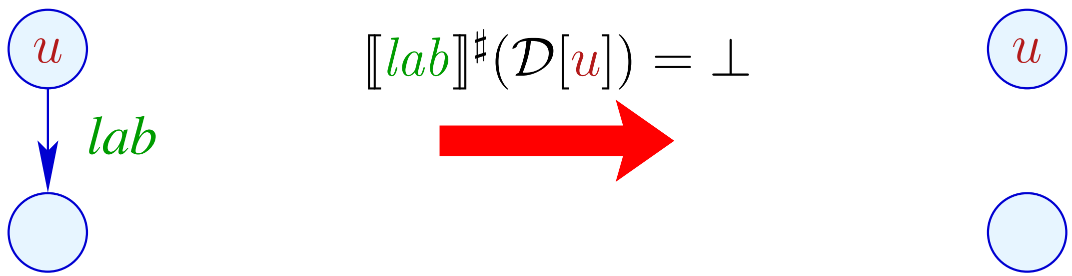
$$(\llbracket \pi \rrbracket (\rho, \mu)) \; \Delta \; (\mathcal{D}[v])$$

whenever $\llbracket \pi \rrbracket (\rho, \mu)$ is defined

Transformation CF: Removal of Dead Code



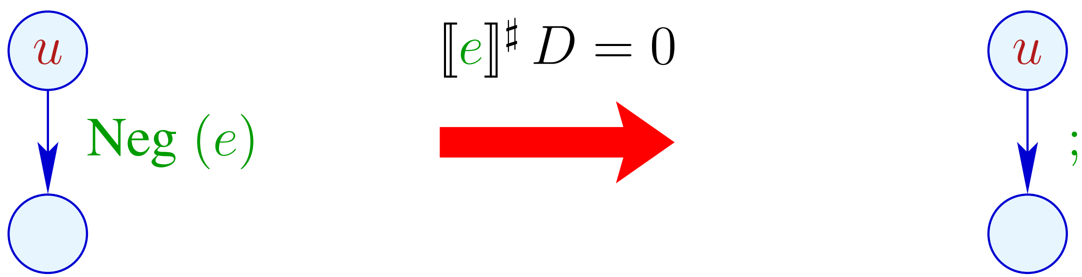
$$\mathcal{D}[u] = \bot$$



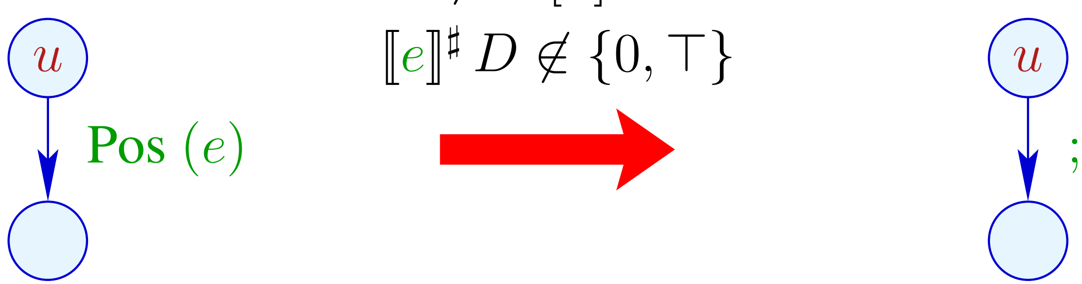
$$[\![lab]\!]^\sharp(\mathcal{D}[u]) = \bot$$

Transformation CF (cont.): Removal of Dead Code

$$\bot \neq \mathcal{D}[u] = D \qquad [\![e]\!]^\sharp \, D = 0$$



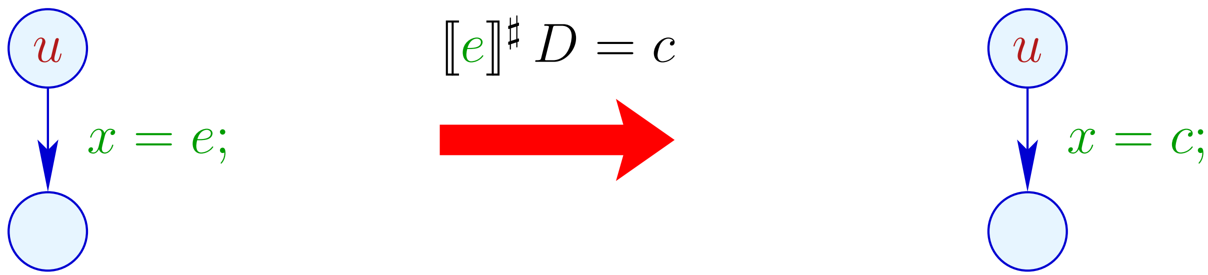
$$\bot \neq \mathcal{D}[u] = D \qquad [\![e]\!]^\sharp \, D \not\in \{0, \top\}$$



Transformation CF (cont.): Simplified Expressions

$$\bot \neq \mathcal{D}[u] = D$$

$$[\![e]\!]^\sharp\, D = c$$



Extensions:

- Instead of complete right-hand sides, subexpressions could be simplified:

$$x + (3 * y) \quad \xrightarrow{\{x \mapsto \top, y \mapsto 5\}} \quad x + 15$$

... and further simplifications be applied, e.g.:

$$\begin{aligned} x * 0 &\Longrightarrow 0 \\ x * 1 &\Longrightarrow x \\ x + 0 &\Longrightarrow x \\ x - 0 &\Longrightarrow x \\ &\ldots \end{aligned}$$

- So far, the information of conditions has not yet be optimally exploited:

$$\mathsf{if}\ (x == 7)$$
$$y = x + 3;$$

Even if the value of x before the if statement is unknown, we at least know that x definitely has the value 7 — whenever the then-part is entered

Therefore, we can define:

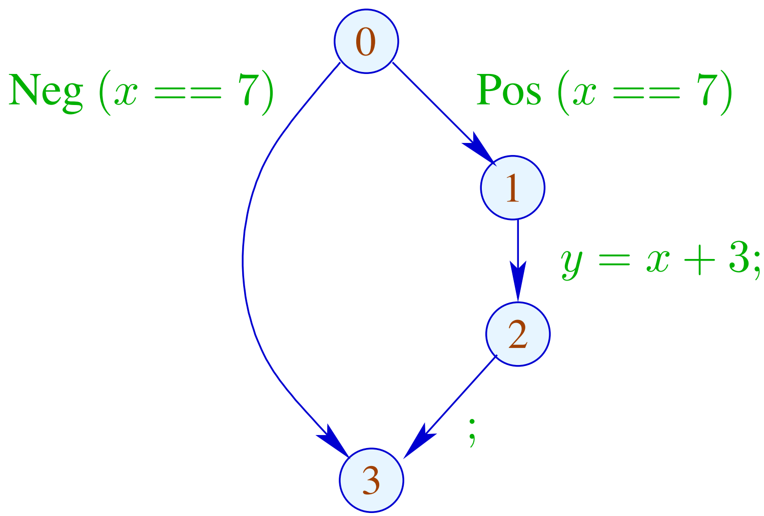
$$[\![\text{Pos}\,(x == e)]\!]^\sharp\, D = \begin{cases} D & \text{if} \quad [\![x == e]\!]^\sharp\, D = 1 \\ \bot & \text{if} \quad [\![x == e]\!]^\sharp\, D = 0 \\ D_1 & \text{otherwise} \end{cases}$$

where

$$D_1 = D \oplus \{x \mapsto (D\, x \sqcap [\![e]\!]^\sharp\, D)\}$$

The effect of an edge labeled $\text{Neg}\,(x \neq e)$ is analogous

Our Example:



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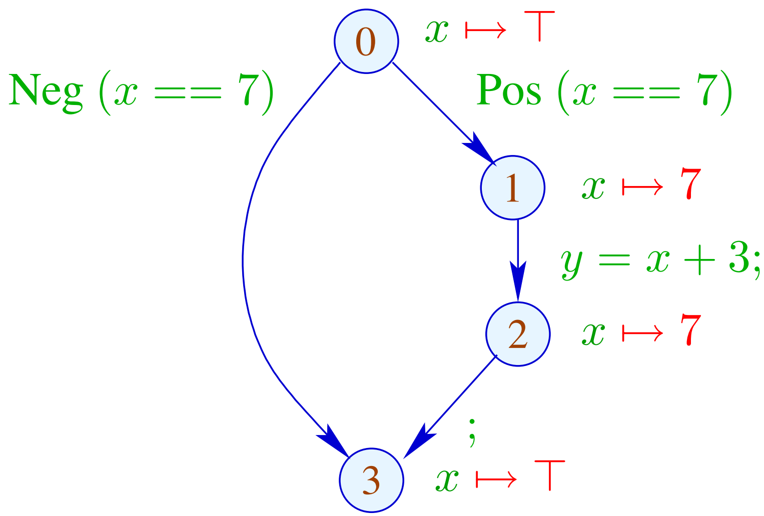

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