A Non-Topological View of Dcpos as Convergence Spaces

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Abstract

The category TOP of topological spaces is not cartesian closed, but can be embedded into the cartesian closed category CONV of *convergence spaces*. It is well-known that the category DCPO of dcpos and Scott continuous functions can be embedded into TOP, and so into CONV, by considering the Scott topology. We propose a different, "*cotopological*" embedding of DCPO into CONV, which, in contrast to the topological embedding, preserves products. If X is a cotopological dcpo, i.e. a dcpo with the cotopological CONV-structure, and Y is a topological space, then $[X \to Y]$ is again topological, and conversely, if X is a topological space, and Y a cotopological complete lattice, then $[X \to Y]$ is again a cotopological complete lattice. For a dcpo D, the topological and the cotopological convergence structures coincide if and only if D is a continuous dcpo. Moreover, cotopological dcpos still enjoy some of the properties which characterise continuous dcpos. For instance, all cotopological complete lattices are injective spaces (in CONV) w.r.t. topological subspace embeddings.

1 Introduction

It is well-known that the category DCPO of dcpos and Scott continuous functions can be embedded into TOP, the category of topological spaces and continuous functions, by endowing each dcpo D with its Scott topology, leading to the topological space D_{s}. This embedding hinges on the fact that a function between dcpos is Scott continuous (i.e., preserves directed joins) if and only if it is continuous w.r.t. the Scott topologies (i.e., the inverse images of Scott open sets are Scott open).

This embedding provides a useful way to look at dcpos as topological spaces, yet it has its drawbacks. For instance, it does not preserve products, i.e., the Scott topology of a product dcpo is not necessarily the same as the product topology derived from the two Scott topologies (in short, $(D \times E)_{\mathrm{s}} = D_{\mathrm{s}} \times E_{\mathrm{s}}$ does not generally hold); see the discussion in [5, page 106]. There are even complete lattices L such that $(L \times L)_{\mathrm{s}} \neq L_{\mathrm{s}} \times L_{\mathrm{s}}$.

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Connected with this product problem is a problem about binary joins in complete lattices. Binary join $\vee : L \times L \to L$ is obviously Scott continuous, and therefore continuous in the sense $(L \times L)_{\mathrm{s}} \to L_{\mathrm{s}}$. Yet it is not always continuous in the proper topological sense, i.e., as a function $L_{\mathrm{s}} \times L_{\mathrm{s}} \to L_{\mathrm{s}}$.

There is a similar problem with pointwise join of functions. While the pointwise join of a directed set of continuous functions is continuous again, this does not hold for the pointwise join of two functions: there are continuous functions $f, g : X \to L_{\mathrm{s}}$ such that their pointwise join $f \vee g : X \to L_{\mathrm{s}}$ is not continuous (in [9, 8], we had to work around this problem by restricting attention to those X where $f \vee g$ is continuous again).

A concrete example where all these problems occur is the complete lattice L constructed in [13] as an example of a complete lattice which is not sober in its Scott topology. If $\vee : L_{\mathrm{s}} \times L_{\mathrm{s}} \to L_{\mathrm{s}}$ were continuous, then L_{s} would be sober by a result in the Compendium [5, Cor. II-1.12]. If $(L \times L)_{\mathrm{s}}$ were equal to $L_{\mathrm{s}} \times L_{\mathrm{s}}$, then $\vee : L_{\mathrm{s}} \times L_{\mathrm{s}} \to L_{\mathrm{s}}$ would be continuous as a Scott continuous function. Finally, with $X = L_{\mathrm{s}} \times L_{\mathrm{s}}$, we have two continuous functions $X \to L_{\mathrm{s}}$, namely the two projections, whose pointwise binary join $\vee : X \to L_{\mathrm{s}}$ is not continuous.

The problems listed above are not very well-known because they do not occur for continuous dcpos (cf. II-4.12 and II-4.13 in the Compendium [5]). Yet they can be avoided altogether by considering a different embedding of DCPO into a topological category—not quite TOP itself, but the larger category CONV of *convergence spaces* [18] (also known as *filter spaces* [12]), whose objects are characterised by the convergence properties of filters.

Every topological space carries a notion of filter convergence which leads to an embedding of TOP as a reflective full subcategory into CONV. Moreover, CONV is cartesian closed in contrast to TOP, i.e., it provides a function space construction such that $[X \times Y \to Z]$ and $[X \to [Y \to Z]]$ are naturally isomorphic, and λ-calculus can be interpreted in the category.

In this paper, we propose a new embedding $(-)_{\mathrm{c}}$ of DCPO into CONV, which, in contrast to the topological embedding $(-)_{\mathrm{s}}$, preserves products and avoids all the problems listed above: We have $(D \times E)_{\mathrm{c}} = D_{\mathrm{c}} \times E_{\mathrm{c}}$ for all dcpos D and E, $\vee : L_{\mathrm{c}} \times L_{\mathrm{c}} \to L_{\mathrm{c}}$ is continuous for all complete lattices L, and pointwise joins of continuous functions $X \to L_{\mathrm{c}}$ are continuous again. The price for this is that D_{c} is not always topological; we shall see that D_{c} is topological (i.e., is an object of the full reflective subcategory TOP of CONV) if and only if $D_{\mathrm{c}} = D_{\mathrm{s}}$, if and only if D is a continuous dcpo. (This gives a new proof that continuous dcpos are well-behaved w.r.t. $(-)_{\mathrm{s}}$.)

The convergence spaces D_{c}, which we call *cotopological dcpos*, exhibit an interesting behaviour in the function space construction:

- If $X = D_{\mathrm{c}}$ is a cotopological dcpo and Y is topological, then $[X \to Y]$ is topological.
- If X is topological and $Y = L_{\mathrm{c}}$ is a cotopological complete lattice, then $[X \to Y]$ is a cotopological complete lattice again.

These properties were the reason for choosing the name "cotopological".

As indicated above, a dcpo D is continuous iff D_{c} is topological, or shortly, continuous = topological + cotopological. Indeed, the cotopological dcpos (lattices) still enjoy many properties familiar from continuous dcpos (lattices). For instance, it is well-known that continuous lattices are *injective spaces* w.r.t. topological embeddings [5, Section II-3]. Here, we show that L_c is injective w.r.t. topological embeddings for any complete lattice L whatsoever.

We start out by a quick recap of filters (Section 2) and convergence spaces (Section 3). There is not much new in there, and most proofs are omitted. In Section 4, we rule out some ugly convergence spaces by imposing certain "niceness conditions" which are obeyed by topological spaces and preserved by product, subspace, and exponentiation. Then we consider *d-spaces* in Section 5, which are spaces whose structure is similar to that of dcpos. In Section 6, D_s is identified as the strongest *topological* d-space structure on D, while D_c is introduced as the strongest d-space structure of all. The final, quite large Section 7 is devoted to prove the main properties of cotopological dcpos (or lattices), i.e., the properties that have been presented in this introduction, and a few more.

2 Filters

2.1 The Lattice of Filters

A *filter* $\mathcal{F}$ on a set X is a subset of the powerset $\mathcal{P}X$ of X which is closed under finite intersection (in particular contains X) and extension to supersets.

(1) If $A \in \mathcal{F}$ and $A \subseteq B$, then $B \in \mathcal{F}$;

(2) $X \in \mathcal{F}$; (3) if A and B are in $\mathcal{F}$, then so is $A \cap B$.

The set of all filters on X is denoted by ΦX.

Arbitrary intersections of filters are filters, so ΦX forms a complete lattice when ordered by inclusion '$\subseteq$'. Besides, directed unions of filters are filters. The bottom element of $(\Phi X, \subseteq)$ is $\{X\}$, while the top element is the *improper filter* $\mathcal{P}X$, the (unique) filter containing the empty set. Since filters are ideals in $(\mathcal{P}X, \supseteq)$, $(\Phi X, \subseteq)$ is an algebraic lattice.

2.2 "Inner Ordering"

If one is more interested in the sets which are in a filter than in the filter as a whole, then it is more natural to order filters as follows [18]:

$$\mathcal{A} \leq_i \mathcal{B} \iff \forall B \in \mathcal{B}\, \exists A \in \mathcal{A} : A \subseteq B.$$

Actually, $\mathcal{A} \leq_i \mathcal{B}$ is equivalent to $\mathcal{A} \supseteq \mathcal{B}$, so '$\leq_i$' is exactly the opposite of '$\subseteq$'. The lattice $(\Phi X, \leq_i)$ will be denoted by $\Phi^i X$.

A *filter base* on X is a downward directed set of subsets of X. Each filter base $\mathcal{B}$ generates a filter $[\mathcal{B}] = \{A \subseteq X \mid A \supseteq B \text{ for some } B \in \mathcal{B}\}$. If $\mathcal{B}$ is already a filter, then $[\mathcal{B}] = \mathcal{B}$. The ordering '$\leq_i$' can be characterised via filter bases:

$$[\mathcal{A}] \leq_i [\mathcal{B}] \iff \forall B \in \mathcal{B}\, \exists A \in \mathcal{A} : A \subseteq B.$$

Indeed, one could introduce '$\leq_i$' as a preorder on filter bases, and define filters as equivalence classes w.r.t. this preorder.

In the following, $[\{\ldots\}]$ is usually abbreviated by $[\ldots]$. Meets and joins w.r.t. '$\leq_{\mathrm{i}}$' will be denoted by '$\wedge$' and '$\vee$'.

(1) Since '$\leq_{\mathrm{i}}$' is the opposite of '$\subseteq$', joins are intersections: $\bigvee_{i\in I}\mathcal{A}_i = \bigcap_{i\in I}\mathcal{A}_i$.

(2) Alternatively, binary joins are given by $[\mathcal{A}]\vee[\mathcal{B}] = [A\cup B \mid A\in\mathcal{A}, B\in\mathcal{B}]$. This does not depend on the choice of the two bases.

(3) Generalising (2), arbitrary joins are given as $\bigvee_{i\in I}[\mathcal{B}_i] = [\bigcup_{i\in I} B_i \mid (B_i)_{i\in I}\in\prod_{i\in I}\mathcal{B}_i]$.

(4) Binary meet is $[\mathcal{A}]\wedge[\mathcal{B}] = [A\cap B \mid A\in\mathcal{A}, B\in\mathcal{B}]$. Unfortunately, this does not generalise to arbitrary meets, and does not correspond to binary union of filters, which in general does not yield a filter again.

(5) *Filtered* meets are given by directed unions or $\bigwedge_{i\in I}[\mathcal{B}_i] = [B \mid B\in\mathcal{B}_i \text{ for some } i\in I]$.

(6) Arbitrary meets are hence given as
$\bigwedge_{i\in I}[\mathcal{B}_i] = [\bigcap_{i\in F} B_i \mid (B_i)_{i\in F}\in\prod_{i\in F}\mathcal{B}_i \text{ for some } F\subseteq_{\mathrm{fin}} I]$.

The lattice $\Phi^{\mathrm{i}}X$ is finitely distributive, but there are examples for $\mathcal{A}\wedge\bigvee_{i\in I}\mathcal{B}_i \neq \bigvee_{i\in I}(\mathcal{A}\wedge\mathcal{B}_i)$.

2.3 Principal Filters

For $A\subseteq X$, $\{A\}$ is a filter base. We abbreviate $[\{A\}]$ by $[A]$; this is usually called a *principal filter* (one might also call it a *set filter*). For $x, x_1, \ldots, x_n \in X$, we further abbreviate $[\{x_1,\ldots,x_n\}]$ by $[x_1,\ldots,x_n]$, and in particular, $[\{x\}]$ by $[x]$, and $[\emptyset]$ by $[\,]$.

Note that $\mathcal{A}\leq_{\mathrm{i}}[B]$ iff $B\in\mathcal{A}$, and $[A]\leq_{\mathrm{i}}[B]$ iff $A\subseteq B$. Further, $\bigvee_{i\in I}[A_i] = [\bigcup_{i\in I} A_i]$, whence $[x_1,\ldots,x_n] = [x_1]\vee\cdots\vee[x_n]$, and $[A]\wedge[B] = [A\cap B]$, and finally, $[\,]$ and $[X]$ are bottom and top in $\Phi^{\mathrm{i}}X$, respectively. Thus, $[-] : \mathcal{P}X\to\Phi^{\mathrm{i}}X$ is an order embedding which preserves arbitrary joins and finite meets (but not infinite meets). This is the main advantage of the "inner view": filters on X can be considered as generalised subsets of X, and we shall see that many properties familiar from $\mathcal{P}X$ carry over to $\Phi^{\mathrm{i}}X$.

2.4 Filters and Functions

A function $f : X\to Y$ induces two functions on subsets: $f^+ : \mathcal{P}X\to\mathcal{P}Y$ with $f^+A = \{fa \mid a\in A\}$ for $A\subseteq X$, and $f^- : \mathcal{P}Y\to\mathcal{P}X$ with $f^-B = \{a\in X \mid fa\in B\}$ for $B\subseteq Y$. These functions are adjoints, i.e., $f^+A\subseteq B \iff A\subseteq f^-B$, and so f^+ preserves all joins and f^- all meets. In addition, f^- preserves all joins as well.

Both functions can be extended to $f^+ : \Phi X\to\Phi Y$ and $f^- : \Phi Y\to\Phi X$ in the obvious way: $f^+[\mathcal{A}] = [f^+A \mid A\in\mathcal{A}]$ and $f^-[\mathcal{B}] = [f^-B \mid B\in\mathcal{B}]$. Then $f^+[A] = [f^+A]$ for $A\subseteq X$, hence $f^+[\,] = [\,]$ and $f^+[x] = [fx]$ for x in X. The assignment $f\mapsto f^+$ is functorial.

These extensions are still adjoints, i.e., $f^+\mathcal{A}\leq_{\mathrm{i}}\mathcal{B} \iff \mathcal{A}\leq_{\mathrm{i}} f^-\mathcal{B}$, and so f^+ preserves all joins and f^- all meets. As in the set case, f^- preserves all joins as well, and unlike the set case, f^+ preserves filtered meets. Using the adjoint property, the set $f^+\mathcal{A}$ can be characterised as follows: $B\in f^+\mathcal{A} \Leftrightarrow f^+\mathcal{A}\leq_{\mathrm{i}}[B] \Leftrightarrow \mathcal{A}\leq_{\mathrm{i}}[f^-B] \Leftrightarrow f^-B\in\mathcal{A}$.

2.5 Product of Filters

For $\mathcal{A}$ in ΦX and $\mathcal{B} \in \Phi Y$, let $\mathcal{A} \times \mathcal{B} = [A \times B \mid A \in \mathcal{A}, B \in \mathcal{B}] \in \Phi(X \times Y)$. Then $[A] \times [B] = [A \times B]$, whence in particular $[x] \times [y] = [(x, y)]$. Further, $[\,] \times \mathcal{B} = \mathcal{A} \times [\,] = [\,]$; and $\mathcal{A} \times \mathcal{B} \neq [\,]$ for $\mathcal{A}, \mathcal{B} \neq [\,]$. There are more properties familiar from sets (where π_1 and π_2 are the projections): $\pi_1^+(\mathcal{A} \times \mathcal{B}) \leq_{\mathrm{i}} \mathcal{A}$ with '=' if $\mathcal{B} \neq [\,]$; the dual property with π_2; $\mathcal{C} \leq_{\mathrm{i}} \pi_1^+\mathcal{C} \times \pi_2^+\mathcal{C}$; and $(f \times g)^+(\mathcal{A} \times \mathcal{B}) = f^+\mathcal{A} \times g^+\mathcal{B}$. Furthermore, '$\times$' distributes over finite joins (but not over infinite ones!).

3 Convergence Spaces

3.1 Definition

There are several notions of convergence spaces in the literature, and worse, there are several names for the same thing: some authors prefer the name *filter spaces* [12, 11], while others use the name *convergence spaces* [18, 2, 14]. Our definition below corresponds to the convergence spaces of [18, 2] and the filter spaces of [12], while the convergence spaces of [14] and the filter spaces of [11] form a smaller class.

Convergence spaces are characterised by specifying which filters converge to which points. Formally, a *convergence space* is a set X together with a relation '$\downarrow$' between ΦX and X such that $[x] \downarrow x$ holds for all x in X (point filter axiom), and $\mathcal{A} \downarrow x$ and $\mathcal{B} \leq_{\mathrm{i}} \mathcal{A}$ (i.e., $\mathcal{B} \supseteq \mathcal{A}$) implies $\mathcal{B} \downarrow x$ (subfilter axiom). (See Section 4 for potential further axioms.) A function $f : X \to Y$ between two convergence spaces is *continuous* if $\mathcal{A} \downarrow x$ implies $f^+\mathcal{A} \downarrow fx$. The category of convergence spaces and continuous functions is called CONV. Note that all constant functions are continuous because of the point filter axiom.

$\mathcal{A} \downarrow x$ is usually read as '$\mathcal{A}$ converges to x', or 'x is a limit of $\mathcal{A}$'. Thus, the relation '$\downarrow$' is called the *convergence relation* or *convergence structure* of the convergence space. A filter has many different limits in general; the set $\{x \in X \mid \mathcal{A} \downarrow x\}$ of all limit points of $\mathcal{A}$ is denoted by $\mathrm{Lim}\,\mathcal{A}$. In particular, the conditions for convergence spaces imply that the improper filter $[\,]$ converges to every x in X. Usually, the improper filter is omitted, but it does not cause any harm in the definition of the category because $f^+[\,] = [\,]$, and so $f^+[\,] \downarrow fx$ is guaranteed for any f.

If $\downarrow_1$ and $\downarrow_2$ are two convergence structures on the same set X, we say $\downarrow_1$ is *stronger* than $\downarrow_2$ and $\downarrow_2$ is *weaker* than $\downarrow_1$ if the identity function $(X, \downarrow_1) \to (X, \downarrow_2)$ is continuous, i.e., if $\mathcal{A} \downarrow_1 x \Rightarrow \mathcal{A} \downarrow_2 x$ (the definition in terms of continuity is in accordance with topology). The strongest convergence structure on a set X is the *discrete structure* with $\mathcal{A} \downarrow x$ iff $\mathcal{A} \leq_{\mathrm{i}} [x]$, and the weakest structure is the *indiscrete structure* where every filter converges to every point. If X is discrete, all functions $f : X \to Y$ are continuous, and likewise for indiscrete Y.

In so far as no confusion can result, we follow the custom of topology using the name of the underlying set X as a shorthand for the convergence space $(X, \downarrow_X)$, and using the same symbol '$\downarrow$' for the convergence relations of all spaces.

3.2 Initial Constructions

Similar to the initial topology for a family of functions, there is an initial convergence structure. Let X be a set, $(Y_i)_{i\in I}$ a family of convergence spaces, and $(f_i : X \to Y_i)_{i\in I}$ a family of (arbitrary) functions. The *initial convergence structure* '$\downarrow$' on X is defined by $\mathcal{A} \downarrow x$ iff $f_i^+\mathcal{A} \downarrow f_i x$ for all i in I (check that the two axioms are satisfied). The universal property of the initial construction is that for all convergence spaces Z and all functions $g : Z \to X$, g is continuous if and only if for all i in I, the compositions $f_i \circ g : Z \to Y_i$ are continuous.

The *product* of a family $(X_i)_{i\in I}$ of convergence spaces is the set $\prod_{i\in I} X_i$ with the initial structure for the projections $\pi_i : \prod_{i\in I} X_i \to X_i$. Hence $\mathcal{A} \downarrow x$ in the product iff $\pi_i^+\mathcal{A} \downarrow x_i$ for all i in I. Note that $\mathcal{A} \downarrow x$ in X and $\mathcal{B} \downarrow y$ in Y implies $\mathcal{A} \times \mathcal{B} \downarrow (x, y)$ in $X \times Y$.

If X is a subset of the convergence space Y, then X with the initial structure induced by the inclusion map $e : X \to Y$ is called a *subspace* of Y. By this definition, e becomes continuous, and moreover, for any convergence space Z and any $f : Z \to X$, f is continuous if and only if $e \circ f : Z \to Y$ is continuous.

The subspace structure is characterised by $\mathcal{A} \downarrow x$ in X iff $e^+\mathcal{A} \downarrow ex$ in Y. A function $e : X \to Y$ with this property is called *initial* or a *pre-embedding*; in this case X is called a *pre-subspace* of Y. Injective pre-embeddings are called *embeddings*. If $e : X \to Y$ is an embedding, then X is isomorphic to the subspace e^+X of Y, and we may call X a subspace of Y as well.

A special case of the subspace construction is the construction of the *equaliser* of continuous $f, g : X \to Y$ as the subspace $\{x \in X \mid fx = gx\}$ of X.

3.3 Function Space

For two convergence spaces X and Y, the *function space* $[X \to Y] = Y^X$ is the set of continuous functions from X to Y with $\mathcal{F} \downarrow f$ iff for all $\mathcal{A} \downarrow x$ in X, $\mathcal{F} \cdot \mathcal{A} \downarrow fx$ holds in Y. Here, $\mathcal{F} \cdot \mathcal{A}$ is $[F \cdot A \mid F \in \mathcal{F},\ A \in \mathcal{A}]$, where $F \cdot A = \{fa \mid f \in F,\ a \in A\}$. Alternatively, $\mathcal{F} \cdot \mathcal{A}$ can be understood as $E^+(\mathcal{F} \times \mathcal{A})$ where $E : [X \to Y] \times X \to Y$ is the evaluation map.

With this function space, CONV becomes a cartesian closed category, and therefore all closed lambda expressions denote continuous functions. This implies in particular that for each x in X, the function $@_x = \lambda f.\, fx$ from $[X \to Y]$ to Y is continuous. Yet the function space is *not* initial for the family $(@_x)_{x\in X}$.

Composition $\circ : [Y \to Z] \times [X \to Y] \to [X \to Z]$ is continuous. For continuous $f : Y \to Z$, $f^X : Y^X \to Z^X$ with $f^X(g) = f \circ g$ is continuous, and this operation preserves initial constructions [10]: if Y is initial for $(f_i : Y \to Z_i)_{i\in I}$, then Y^X is initial for $(f_i^X : Y^X \to Z_i^X)_{i\in I}$. In particular, if $e : Y \hookrightarrow Z$ is a (pre-)embedding, then so is $e^X : [X \to Y] \hookrightarrow [X \to Z]$, and $\prod_{i\in I}[X \to Y_i] \cong [X \to \prod_{i\in I} Y_i]$ holds.

3.4 Topological Spaces as Convergence Spaces

In a topological space $(X, \mathcal{O})$, a filter $\mathcal{A} \in \Phi X$ converges to x in X if $\mathcal{A}$ contains all opens that contain x. This can be expressed differently: A set $N \subseteq X$ is a *neighbourhood* of a point x of X if there is some open O in $\mathcal{O}$ such that $x \in O \subseteq N$. The collection $\mathcal{N}(x)$ of all neighbourhoods of x is a filter, and the above definition of convergence amounts to saying $\mathcal{A} \downarrow_{\mathcal{O}} x$ iff $\mathcal{A} \supseteq \mathcal{N}(x)$ iff $\mathcal{A} \leq_{\mathrm{i}} \mathcal{N}(x)$. Clearly, the two convergence space axioms are satisfied. Note that the discrete topology yields the discrete convergence structure, and likewise for the indiscrete case.

A function $f : (X, \mathcal{O}) \to (Y, \mathcal{O}')$ is continuous in the topological sense if and only if $f : (X, \downarrow_{\mathcal{O}}) \to (Y, \downarrow_{\mathcal{O}'})$ is continuous in the convergence space sense. Thus, the construction $(X, \mathcal{O}) \mapsto (X, \downarrow_{\mathcal{O}})$ is the object part of a full and faithful functor $\mathrm{C} : \mathrm{TOP} \to \mathrm{CONV}$, and TOP can be considered as a full subcategory of CONV (the *topological* convergence spaces). This subcategory is closed under initial constructions, but not under function space (otherwise TOP would be cartesian closed). If X is a set, $(Y_i)_{i \in I}$ a family of topological spaces, and $(f_i : X \to Y_i)_{i \in I}$ a family of (arbitrary) functions, then it does not matter whether the initial construction in CONV is applied to the spaces $\mathrm{C}Y_i$, or whether C is applied to the result of the initial construction in TOP; the final result is the same in both cases.

Thus products and (pre-)subspaces of topological convergence spaces are again topological. Pre-embeddings $e : X \to Y$ between topological spaces are characterised by the property that each open U of X is of the form $e^{-}V$ for some open V of Y.

In the sequel, X and $\mathrm{C}X$ will often be identified. A particular example is Sierpinski space $\Omega = \{0, 1\}$ where all filters converge to 0, while $[1]$ is the only proper filter converging to 1, i.e., $\mathcal{B} \downarrow 1$ iff $\mathcal{B} \leq_{\mathrm{i}} [1]$, iff $\{1\} \in \mathcal{B}$.

3.5 The Induced Topology

Using Sierpinski space, we can define a topology on (the carrier set of) a convergence space X as follows: A subset O of X is *open* iff its characteristic function $\chi_O : X \to \Omega$ is continuous. This is equivalent to $\mathcal{A} \downarrow x \Rightarrow \chi_O^{+}\mathcal{A} \downarrow \chi_O x$. By the characterisation of convergence in Ω, we may restrict to the case $\chi_O x = 1$, or $x \in O$. Thus, χ_O is continuous iff $\mathcal{A} \downarrow x \in O$ implies $\chi_O^{+}\mathcal{A} \downarrow 1$. The latter means $\{1\} \in \chi_O^{+}\mathcal{A}$, or $O = \chi_O^{-}\{1\} \in \mathcal{A}$. Thus we obtain that O is open iff $\mathcal{A} \downarrow x \in O$ implies $O \in \mathcal{A}$.

Arbitrary unions and finite intersections of opens are open, so we get indeed a topology on X, the *induced topology*. When we speak of open or closed subsets of a convergence space, this always refers to the induced topology. By the definition of open sets, $\mathcal{A} \downarrow x$ always implies $\mathcal{A} \leq_{\mathrm{i}} \mathcal{N}(x)$ where $\mathcal{N}(x)$ is the neighbourhood filter of x in the induced topology. If X is a topological space, the induced topology of $\mathrm{C}X$ is the original topology so that no confusion can arise, and $\mathcal{A} \downarrow x$ is equivalent to $\mathcal{A} \leq_{\mathrm{i}} \mathcal{N}(x)$.

Let $\mathrm{T}X$ be the topological space with the induced topology. If $f : X \to Y$ is continuous in the convergence space sense, then $f^{-}V$ is open for every open set V of Y, and so $f : \mathrm{T}X \to \mathrm{T}Y$ is continuous in the topological sense. The opposite implication does not hold in general, but it holds for topological convergence spaces. More precisely, if X is a convergence space and Y a topological space, then $f : X \to \mathrm{C}Y$ is CONV-continuous if and only if $f : \mathrm{T}X \to Y$ is TOP-continuous, i.e., T is left adjoint to C, and since $\mathrm{T} \circ \mathrm{C} = \mathsf{id}$, TOP is a reflective subcategory of CONV.

Note that in general $\mathrm{T}(X \times Y)$ is different from $\mathrm{T}X \times \mathrm{T}Y$ (the induced topology of $X \times Y$ is not always the product topology; examples will come up later). If U is open in X and V is open in Y, then $U \times V$ is open in $X \times Y$ (because $U \times V = \pi_1^- U \cap \pi_2^- V$), but these sets do not form a basis of the induced topology of $X \times Y$ in general. (As already pointed out, these problems do not occur if X and Y are topological convergence spaces; in this case, $X \times Y$ is again a topological convergence space with the product topology.)

Subspaces suffer from a similar problem. The following finite example was provided by Matias Menni.

Example 3.1 Let $Y = \{-1, 0, 1\}$ with $\mathcal{A} \downarrow -1$ iff $\mathcal{A} \leq_{\mathrm{i}} [-1, 0]$, $\mathcal{A} \downarrow 1$ iff $\mathcal{A} \leq_{\mathrm{i}} [1, 0]$, and finally $\mathcal{A} \downarrow 0$ iff $\mathcal{A} \leq_{\mathrm{i}} [-1, 0, 1]$, i.e., all filters converge to 0. If 1 is in an open set U, then $U \in [1, 0]$ and so $0 \in U$. If 0 is in U, then $U \in [-1, 0, 1]$ and so $U = Y$. Similar arguments hold for -1. Thus, the induced topology of Y is the indiscrete topology (although Y does not have the indiscrete convergence structure). Let X be the subspace $\{-1, 1\}$ of Y (taken in CONV). We get $\mathcal{A} \downarrow_X 1$ iff $\mathcal{A} \leq_{\mathrm{i}} [1]$, and likewise for -1, i.e., X is discrete, and therefore, $\mathrm{T}X$ (discrete) is not a topological subspace of $\mathrm{T}Y$ (indiscrete).

Of course, there are no problems for subspaces of topological convergence spaces.

3.6 The Induced Preorder

The *induced preorder* of a convergence space X is the specialisation preorder of its induced topology, i.e., $x \sqsubseteq y$, iff $x \in \mathsf{cl}\{y\}$, iff y is in every open containing x, iff $px \sqsubseteq py$ for all continuous $p : X \to \Omega$ (where Ω is ordered by $0 \sqsubseteq 1$). When speaking of lower sets, lower bounds, upper sets etc. in a convergence space, we always refer to the induced preorder. As usual, the symbol '$\downarrow$' will be used as a prefix operator for principal ideals $\downarrow a = \{x \mid x \sqsubseteq a\}$ and lower closure $\downarrow A = \bigcup_{a \in A} \downarrow a$. It will always be clear from the context whether '$\downarrow$' is used in this way or to denote a convergence relation.

Continuous functions are monotonic in the induced preorders. Therefore, $x \sqsubseteq x'$ in an initial space X w.r.t. $(f_i : X \to Y_i)_{i \in I}$ implies $f_i x \sqsubseteq f_i x'$ for all i in I, and $f \sqsubseteq g$ in $[X \to Y]$ implies $fx \sqsubseteq gx$ for all x in X. In both cases, the converse does not hold in general. Example 3.1 presents a situation where a subspace preorder (discrete) is different from the restriction of the preorder of the whole space to the subset (indiscrete).

In any space, $[y] \downarrow x$ implies $x \sqsubseteq y$, but the converse does not hold in general. For instance, in the space $Y = \{-1, 0, 1\}$ of Example 3.1, the induced topology is indiscrete, and so the induced preorder is $Y \times Y$. In particular, $-1 \sqsubseteq 1$ holds, but $[1] \downarrow -1$ does not hold.

In Section 4, we shall introduce some classes of convergence spaces which avoid the above-mentioned problems.

3.7 $\mathcal{T}_0$ and $\mathcal{T}_1$

A convergence space is $\mathcal{T}_0$ iff $x \sqsubseteq y$ and $y \sqsubseteq x$ together imply $x = y$ (anti-symmetry of the induced preorder), and $\mathcal{T}_1$ iff $x \sqsubseteq y$ implies $x = y$ (the induced preorder is equality). Clearly, these are properties of the induced topology. Therefore, they are equivalent to the well-known topological notions for topological convergence spaces.

If $(f_i : X \to Y_i)_{i\in I}$ is a point-separating family of continuous functions and all spaces Y_i are $\mathcal{T}_0$ ($\mathcal{T}_1$), then so is X. Here point-separating means that $f_i x = f_i x'$ for all i implies $x = x'$. This includes products and subspaces, but also function spaces because of $(\lambda f.\, fx : [X \to Y] \to Y)_{x\in X}$ (it is not required that X carries the initial structure w.r.t. the family). Thus the separation properties $\mathcal{T}_0$ and $\mathcal{T}_1$ carry over from Y to $[X \to Y]$, for arbitrary X.

4 Niceness Properties

There are quite pathological convergence spaces around, for instance space Y of Example 3.1 whose convergence structure induces the indiscrete topology, but admits non-trivial discrete subspaces. Such pathologies can be ruled out by imposing further conditions on the convergence structure, which we shall call niceness properties (one could also say additional axioms on top of the existing two). Of course, these niceness properties should not destroy anything of what has been outlined above. Therefore, we define that a property N is a niceness property if the following holds:

(1) Every topological convergence space satisfies N.

(2) Property N is preserved by initial constructions (and thus by products, subspaces, and in particular equalisers).

(3) Property N is preserved by exponentiation, i.e., if Y has the property, then $[X \to Y]$ has it as well, no matter whether X satisfies the property or not.

4.1 Merge-Niceness

Recall the subfilter axiom saying that if $\mathcal{A} \downarrow x$ and $\mathcal{A}' \leq_{\mathrm{i}} \mathcal{A}$, then $\mathcal{A}' \downarrow x$ holds as well. Merge-niceness provides a step in the opposite direction:

- If $\mathcal{A} \downarrow x$ and $\mathcal{B} \downarrow x$, then $\mathcal{A} \vee \mathcal{B} \downarrow x$ (i.e., $\mathcal{A} \cap \mathcal{B} \downarrow x$).

As usual, '$\vee$' refers to the "inner view" $\Phi^{\mathrm{i}} X = (\Phi X, \leq_{\mathrm{i}})$.

In topological spaces, $\mathcal{A} \downarrow x$ iff $\mathcal{A} \leq_{\mathrm{i}} \mathcal{N}(x)$, and so merge-niceness is certainly satisfied; even its infinite version holds.

Let X be initial for $(f_i : X \to Y_i)_{i\in I}$ where all Y_i are merge-nice. If $\mathcal{A}, \mathcal{B} \downarrow x$ in X, then $f_i^+\mathcal{A}, f_i^+\mathcal{B} \downarrow f_i x$ for all i, whence $f_i^+(\mathcal{A} \vee \mathcal{B}) = f_i^+\mathcal{A} \vee f_i^+\mathcal{B} \downarrow f_i x$, which gives $\mathcal{A} \vee \mathcal{B} \downarrow x$ by initiality. This argument would be valid for infinite joins as well.

Let Y be merge-nice and $\mathcal{F}_1, \mathcal{F}_2 \downarrow f$ in $[X \to Y]$. Then for all $\mathcal{A} \downarrow x$, $\mathcal{F}_1 \cdot \mathcal{A} \downarrow fx$ and $\mathcal{F}_2 \cdot \mathcal{A} \downarrow fx$, whence by merge-niceness $\mathcal{F}_1 \cdot \mathcal{A} \vee \mathcal{F}_2 \cdot \mathcal{A} \downarrow fx$. This filter is the same as $(\mathcal{F}_1 \vee \mathcal{F}_2) \cdot \mathcal{A}$, and so we are done. This argument does *not* carry over to infinite joins. Remember $\mathcal{F} \cdot \mathcal{A} = E^+(\mathcal{F} \times \mathcal{A})$ where E is evaluation. Unlike the set version of '$\times$', the filter version does not distribute over infinite joins in general.

Merge-nice convergence spaces are sometimes called limit spaces [18, 14]. Some authors include merge-niceness into the definition of the spaces they consider, but it is not needed to obtain a cartesian closed category. For the topic of the paper at hand, it is of minor importance, and worse, many of the "cotopological" convergence spaces considered later do not satisfy it. Merge-niceness on its own does not rule out the pathologies concerned with subspace topology and preorder; for, space Y in Example 3.1 is merge-nice because of the very way its convergence structure has been defined. On the other hand, merge-niceness is needed for the inclusion into Scott's category EQU of equilogical spaces [16, 1] which works smoothly only for merge-nice convergence spaces (see [7] where convergence spaces are called filter spaces).

4.2 Up-Niceness

The induced preorder of a convergence space X gives the usual up-closure $\uparrow A$ for subsets A of X. This up-closure can be extended to filters by defining $\uparrow\mathcal{A} = [\uparrow A \mid A \in \mathcal{A}]$. Note that in $\Phi^{\mathrm{i}}X$, we have $\mathcal{A} \leq_{\mathrm{i}} \uparrow\mathcal{A}$ as it is familiar from sets, '$\uparrow$' is monotonic, and $\uparrow\uparrow\mathcal{A}$ is the same as $\uparrow\mathcal{A}$.

Then up-niceness is the following property:

- If $\mathcal{A} \downarrow x$, then also $\uparrow\mathcal{A} \downarrow x$.

A topological space is up-nice since $\uparrow\mathcal{N}(x) = \mathcal{N}(x)$, and so, $\mathcal{A} \leq_{\mathrm{i}} \mathcal{N}(x)$ iff $\uparrow\mathcal{A} \leq_{\mathrm{i}} \mathcal{N}(x)$. Up-niceness is preserved by initial constructions and function space, as required for a niceness property. For initial constructions, one needs the property $f_i^+(\uparrow\mathcal{A}) \leq_{\mathrm{i}} \uparrow f_i^+\mathcal{A}$ which holds due to monotonicity of f_i. For function space, one needs $(\uparrow\mathcal{F}) \cdot \mathcal{A} \leq_{\mathrm{i}} \uparrow(\mathcal{F} \cdot \mathcal{A})$ which holds because the corresponding property for sets holds, and ultimately, since $g \sqsupseteq f$ implies $ga \sqsupseteq fa$ for all a.

In up-nice convergence spaces, the limit points of principal filters can be completely characterised:

Proposition 4.1
Let X be an up-nice space, and $A \subseteq X$. Then $[A] \downarrow x$ iff x is a lower bound of A.

Proof: For every a in A, $[a] \leq_{\mathrm{i}} [A]$ holds. Hence, $[A] \downarrow x$ implies $[a] \downarrow x$ for all a in A by the subfilter axiom, and thus x is a lower bound of A. Conversely, if x is a lower bound of A, then $A \subseteq \uparrow x$, whence $[A] \leq_{\mathrm{i}} [\uparrow x] = \uparrow[x]$, and the latter converges to x because of up-niceness and the point filter axiom. □

Hence, finite up-nice spaces are topological. (All filters are principal, and $[A] \downarrow x$ iff $[A] \leq_{\mathrm{i}} [\uparrow x] = \mathcal{N}(x)$, the neighbourhood filter in the Alexandroff topology.)

From the above characterisation of the limits of principal filters, $[y] \downarrow x \iff x \sqsubseteq y$ follows. This property suffices to conclude that the induced preorder of initial up-nice spaces is well-behaved: $x \sqsubseteq x'$ implies $f_i x \sqsubseteq f_i x'$ for all i, which gives $f_i^+[x'] = [f_i x'] \downarrow f_i x$, and thus $[x'] \downarrow x$ by initiality, which finally implies $x \sqsubseteq x'$ showing that all these statements are equivalent. Therefore, the preorder of products of up-nice spaces is componentwise, and the preorder of a subspace of an up-nice space is obtained by restriction.

Moreover, up-niceness implies that the preorder in function spaces is pointwise: If $fx \sqsubseteq gx$ for all x, then $g^+A \subseteq \uparrow f^+A$ holds for all subsets, which carries over to filters. Using this relation, $[g] \downarrow f$ can be proved: if $\mathcal{A} \downarrow x$, then $[g] \cdot \mathcal{A} = g^+\mathcal{A} \downarrow fx$ because $g^+\mathcal{A} \leq_{\mathrm{i}} \uparrow f^+\mathcal{A}$ and $\uparrow f^+\mathcal{A} \downarrow fx$ by continuity of f and up-niceness.

4.3 Down-Niceness

While the previous properties dealt with the filters converging to a fixed point, the properties that follow are statements about the set of limit points of a fixed filter. Down-niceness states that it is a lower set:

- If $\mathcal{A} \downarrow y$ and $y \sqsupseteq x$, then $\mathcal{A} \downarrow x$.

By definition, $y \sqsupseteq x$ means $\mathcal{N}(x) \subseteq \mathcal{N}(y)$, or $\mathcal{N}(y) \leq_{\mathrm{i}} \mathcal{N}(x)$, where $\mathcal{N}(x)$ is the neighbourhood filter of x. From this, it is immediate that topological spaces are down-nice. For initial constructions, $\mathcal{A} \downarrow y \sqsupseteq x$ implies $f_i^+\mathcal{A} \downarrow f_i y \sqsupseteq f_i x$ for all i, whence $f_i^+\mathcal{A} \downarrow f_i x$ for all i, and thus $\mathcal{A} \downarrow x$. If $\mathcal{F} \downarrow f \sqsupseteq g$ in a function space, then $\mathcal{F} \cdot \mathcal{A} \downarrow fx \sqsupseteq gx$ for all $\mathcal{A} \downarrow x$, whence $\mathcal{F} \cdot \mathcal{A} \downarrow gx$ for all $\mathcal{A} \downarrow x$, and thus $\mathcal{F} \downarrow g$.

In presence of down-niceness, the following three statements are equivalent:

(1) $x \sqsubseteq y$; (2) $[y] \downarrow x$; (3) for all filters $\mathcal{A}$, $\mathcal{A} \downarrow y$ implies $\mathcal{A} \downarrow x$.

Here, (1) $\Rightarrow$ (3) is down-niceness, while (3) $\Rightarrow$ (2) and (2) $\Rightarrow$ (1) always hold. From the equivalence of (1) and (2), it follows as in up-nice spaces that the preorder in initial constructions is well-behaved, i.e., $x \sqsubseteq x'$ iff $f_i x \sqsubseteq f_i x'$ for all i. Furthermore, the preorder is pointwise in function spaces: If $\mathcal{F} \downarrow f$ in a function space and $fx \sqsupseteq gx$ for all x, then $\mathcal{F} \cdot \mathcal{A} \downarrow fx \sqsupseteq gx$ for all $\mathcal{A} \downarrow x$, whence $\mathcal{F} \downarrow g$ follows. By the stated equivalences, $\mathcal{F} \downarrow f \Rightarrow \mathcal{F} \downarrow g$ means $f \sqsupseteq g$.

4.4 Order Niceness

A convergence space is *order-nice* if it is both up-nice and down-nice.

4.5 Closure Niceness

Down-niceness is equivalent to the property that for every filter $\mathcal{A}$, the set $\mathrm{Lim}\,\mathcal{A} = \{x \mid \mathcal{A} \downarrow x\}$ of limit points is a lower set. An obvious strengthening is the following (closure niceness):

- For every filter $\mathcal{A}$, the set $\mathrm{Lim}\,\mathcal{A}$ of limit points is closed (in the induced topology).

To show that topological spaces are closure-nice, let x be in $\mathsf{cl}\,(\mathrm{Lim}\,\mathcal{A})$. Then each open set containing x also contains a limit point of $\mathcal{A}$, and hence is in $\mathcal{A}$. This shows $\mathcal{A} \leq_{\mathrm{i}} \mathcal{N}(x)$, and thus $\mathcal{A} \downarrow x$. For initial structures, $\mathrm{Lim}\,\mathcal{A}$ is $\bigcap_{i \in I} f_i^-(\mathrm{Lim}\,(f_i^+\mathcal{A}))$, and for function spaces, $\mathrm{Lim}\,\mathcal{F} = \bigcap_{\mathcal{A} \downarrow x} (@_x)^-(\mathrm{Lim}\,(\mathcal{F} \cdot \mathcal{A}))$. These are closed sets since the functions f_i and $@_x = \lambda f.\, fx$ are continuous (in CONV and therefore in the induced topologies).

5 d-Spaces and Join Spaces

A topological space is a *d-space* [19, 4] (monotone convergence space in [5]) if its specialisation preorder forms a dcpo, and all open sets are Scott open; or equivalently, if every directed set of points has a least upper bound which is also a limit point of the set. Clearly, this notion captures essential topological properties of dcpos, and for any dcpo D, the Scott topology is the strongest topology which yields a d-space whose induced dcpo is D.

Below, we extend the notion of d-space to CONV in such a way that its restriction to TOP yields the original notion. The cotopological convergence structure on a dcpo D will be the strongest d-space structure whose induced dcpo is D. Hence, all properties of general d-spaces will be inherited by cotopological dcpos.

Join spaces are to complete lattices what d-spaces are to dcpos. They have some additional properties which are inherited by all cotopological complete lattices.

5.1 d-Spaces

Actually, there are several different ways to generalise the topological notion of d-spaces to CONV. Our choice gives good properties, in particular closure under exponentiation.

An order-nice convergence space is a *d-space* if the induced preorder is a dcpo (this includes anti-symmetry), and all limit sets $\mathrm{Lim}\,\mathcal{A}$ are closed under directed joins. (Here, "order-nice" may be relaxed to "up-nice", if "closed under directed joins" is strengthened to "Scott closed".) All finite up-nice $\mathcal{T}_0$ spaces are d-spaces, and all $\mathcal{T}_1$ spaces are d-spaces.

To derive properties of d-spaces, the following definition is useful: For a directed set Δ in a poset D, let $\langle\Delta\rangle = [\uparrow d \mid d \in \Delta]$.

Lemma 5.1

(1) In any $\mathcal{T}_0$ convergence space: If x is an upper bound of Δ and $\langle\Delta\rangle \downarrow x$, then $x = \bigsqcup \Delta$.

(2) In a d-space, $\langle\Delta\rangle \downarrow \bigsqcup \Delta$ holds, and hence the implication in (1) becomes an equivalence.

Proof:

(1) Assume $\langle\Delta\rangle \downarrow x$ and let u be an upper bound of Δ. This means $u \in \uparrow d$ for all d in Δ, whence $[u] \leq_{\mathrm{i}} \langle\Delta\rangle$. Thus, $\langle\Delta\rangle \downarrow x$ implies $[u] \downarrow x$, whence $x \sqsubseteq u$.

(2) For every d in Δ, $\langle\Delta\rangle \leq_{\mathrm{i}} [\uparrow d] = \uparrow[d]$. By up-niceness, $\uparrow[d] \downarrow d$, and so $\langle\Delta\rangle \downarrow d$. Hence, Δ is a subset of $\mathrm{Lim}\,\langle\Delta\rangle$, whence $\langle\Delta\rangle \downarrow \bigsqcup \Delta$ by the d-space property. $\square$

Proposition 5.2 Let X be a d-space, Y an up-nice $\mathcal{T}_0$-space, and $f : X \to Y$ a continuous function. Then for all directed sets $\Delta \subseteq X$, $f(\bigsqcup \Delta) = \bigsqcup f^+\Delta$ holds.

Proof: As a continuous function, f is monotonic, and therefore, $f^+\Delta$ is directed again. By Lemma 5.1 (2), $x = \bigsqcup \Delta$ is an upper bound of Δ, and $\langle\Delta\rangle \downarrow x$ holds. By monotonicity, fx is an upper bound of $f^+\Delta$, and continuity of f and up-niceness of Y together imply $\uparrow f^+\langle\Delta\rangle \downarrow fx$. Now, $\uparrow f^+\langle\Delta\rangle = [\uparrow f^+(\uparrow d) \mid d \in \Delta] = [\uparrow fd \mid d \in \Delta] = \langle f^+\Delta\rangle$, which gives $\langle f^+\Delta\rangle \downarrow fx$. By Lemma 5.1 (1), $fx = \bigsqcup f^+\Delta$ follows. $\square$

Corollary 5.3 All continuous functions between d-spaces are Scott continuous.

Using the d-space Ω and the equivalence between open sets and continuous functions to Ω, we obtain:

Corollary 5.4
In a d-space, all open sets are Scott open, and all closed sets are Scott closed.

This property characterises the d-spaces among up-nice closure-nice spaces: if all closed sets are Scott closed, then in particular $\mathrm{Lim}\,\mathcal{A}$ is Scott closed. Thus, a topological space is a d-space iff its specialisation preorder forms a dcpo and all open sets are Scott open—exactly the topological d-space notion.

Theorem 5.5 Products of d-spaces are d-spaces again.

Proof: Up-niceness guarantees that the induced preorder of the product $X = \prod_{i \in I} X_i$ is the product ordering. By order theory, X is a dcpo in this order. If $\mathcal{A} \downarrow d$ for all d in a directed set Δ, then $\mathcal{A}_i \downarrow d_i$ for all i (where $\mathcal{A}_i$ abbreviates $\pi_i^+ \mathcal{A}$), whence $\mathcal{A}_i \downarrow x_i$ where $x_i = \bigsqcup_{d \in \Delta} d_i$, and finally $\mathcal{A} \downarrow (x_i)_{i \in I} = \bigsqcup \Delta$. $\square$

Proposition 5.6
Subspaces of a d-space that are closed under directed joins are d-spaces again.

Proof: Let X be a subset of the d-space Y, closed under directed joins, and let $e : X \to Y$ be the subspace embedding. If $\mathcal{A} \downarrow d$ for all d in a directed subset Δ of X, then $e^+ \mathcal{A} \downarrow ed$, whence $e^+ \mathcal{A} \downarrow \bigsqcup e^+ \Delta$ by the d-space property of Y. Since X is closed under directed joins, $\bigsqcup e^+ \Delta$ equals $e(\bigsqcup \Delta)$, and thus $\mathcal{A} \downarrow \bigsqcup \Delta$ as required. $\square$

Theorem 5.7 If X is a d-space and Y an up-nice $\mathcal{T}_0$ space, then equalisers of continuous $f, g : X \to Y$ are d-spaces again.

Proof: Let Δ be a directed set in the equaliser. By Prop. 5.2, $f(\bigsqcup \Delta) = \bigsqcup f^+ \Delta = \bigsqcup g^+ \Delta = g(\bigsqcup \Delta)$ holds, and thus $\bigsqcup \Delta$ is in the equaliser again. Therefore, the equaliser is closed under directed joins, and hence a d-space again by Prop. 5.6. $\square$

Proposition 5.8 If Δ is a directed set of continuous functions from an arbitrary space X to a d-space Y, then the function $g = (x \mapsto \bigsqcup_{f \in \Delta} fx)$ is well-defined, continuous, and the join of Δ in $[X \to Y]$.

Proof: The joins in the definition of g are directed, so g is a well-defined function. By up-niceness, the order of the function space is pointwise, and so, g obviously is the join of Δ, provided that it is continuous. For continuity, consider $\mathcal{A} \downarrow x$, whence $\uparrow f^+ \mathcal{A} \downarrow fx$ for all f in Δ by continuity and up-niceness. For all such f, $f \sqsubseteq g$ holds, whence $g^+ A \subseteq \uparrow f^+ A$ for $A \in \mathcal{A}$, and accordingly, $g^+ \mathcal{A} \leq_{\mathrm{i}} \uparrow f^+ \mathcal{A}$. Therefore, we have $g^+ \mathcal{A} \downarrow fx$ for all f in Δ, whence $g^+ \mathcal{A} \downarrow gx$ by the d-space property of Y. $\square$

Theorem 5.9 If Y is a d-space, then $[X \to Y]$ is a d-space for any X.

Proof: By Prop. 5.8, $[X \to Y]$ is a dcpo with pointwise directed joins. If $\mathcal{F} \downarrow f$ for all f in a directed set Δ, then for all $\mathcal{A} \downarrow x$, $\mathcal{F} \cdot \mathcal{A} \downarrow fx$ holds for all f in Δ, whence $\mathcal{F} \cdot \mathcal{A} \downarrow \bigsqcup_{F \in \Delta} fx = (\bigsqcup \Delta)(x)$ by the d-space property of Y and Prop. 5.8. $\square$

5.2 Join Spaces

We now specialise d-spaces to complete lattices. Before we come to the definition, we start with a lemma about binary joins. Let's say $\mathcal{A}$ is an upper filter if $\uparrow\mathcal{A} = \mathcal{A}$, i.e., $\mathcal{A}$ is generated by a filter base of upper sets.

Lemma 5.10 Let P be a poset, where binary joins $x \vee y$ exist for all x, y in P. Then for all upper sets B and C, $\vee^+(B \times C) = B \cap C$ holds, and similarly for upper filters $\mathcal{B}$ and $\mathcal{C}$, we have $\vee^+(\mathcal{B} \times \mathcal{C}) = \mathcal{B} \wedge \mathcal{C}$.

Proof: The set statement is straightforward, and the filter statement follows from it since both sides may be written in terms of the upper sets in appropriate filter bases. □

Theorem 5.11 For an order-nice convergence space X, the following are equivalent:

(1) X is a d-space with a least element 0 and a continuous binary join operator $\vee : X \times X \to X$.

(2) The induced preorder of X is a complete lattice, and the limit sets $\mathrm{Lim}\,\mathcal{A}$ are closed under arbitrary joins.

(3) For every filter $\mathcal{A}$, there is a unique point a such that $\mathrm{Lim}\,\mathcal{A} = \downarrow a$.

Such spaces are called *join spaces*.

Proof: Clearly, a space as in (1) is a complete lattice. By the d-space property, the limit sets are closed under directed joins. They are closed under the empty join, i.e., contain 0, since for each filter $\mathcal{A}$, $\mathcal{A} \leq_{\mathrm{i}} [X] = \uparrow[0]$ holds, and $\uparrow[0] \downarrow 0$ by up-niceness. For closure under binary join, assume $\mathcal{A} \downarrow x_1, x_2$, whence $\uparrow\mathcal{A} \downarrow x_1, x_2$ by up-niceness, and therefore $\uparrow\mathcal{A} = \uparrow\mathcal{A} \wedge \uparrow\mathcal{A} = \vee^+(\uparrow\mathcal{A} \times \uparrow\mathcal{A}) \downarrow x_1 \vee x_2$ by continuity of '$\vee$', whence $\mathcal{A} \downarrow x_1 \vee x_2$. Closure under directed joins, binary joins, and empty join implies closure under all joins by a standard argument.

From (2) and down-niceness, (3) is obvious. For the opposite direction, one has to show that (3) is sufficient to conclude that X is a complete lattice. For any $A \subseteq X$, $\mathrm{Lim}[A]$ is the set of lower bounds of A by up-niceness and Prop. 4.1. Property (3) thus gives the greatest lower bound of A.

For (2) $\Rightarrow$ (1), assume X is a space as in (2). Then clearly X is a d-space with a least element and binary joins. The only thing to show is continuity of '$\vee$'. If $\mathcal{A}_1 \downarrow x_1$ and $\mathcal{A}_2 \downarrow x_2$, then $\uparrow\mathcal{A}_1 \wedge \uparrow\mathcal{A}_2 \downarrow x_1, x_2$ by up-niceness and the subfilter axiom, and so $\vee^+(\uparrow\mathcal{A}_1 \times \uparrow\mathcal{A}_2) \downarrow x_1 \vee x_2$ by Lemma 5.10. Clearly, $\vee^+(\mathcal{A}_1 \times \mathcal{A}_2) \leq_{\mathrm{i}} \vee^+(\uparrow\mathcal{A}_1 \times \uparrow\mathcal{A}_2)$ holds, which concludes the proof. □

Theorem 5.12 Products of join spaces are join spaces again.

Proof: The product $X = \prod_{i \in I} X_i$ is a d-space by Theorem 5.5. Its least element is $(0_i)_{i \in I}$ where 0_i is the least element of X_i. Binary join is componentwise; its continuity can be shown using the universal property of products. □

Theorem 5.13 If Y is a join space, then so is $[X \to Y]$ for any X. Joins in $[X \to Y]$ are pointwise: $(\bigvee_{i \in I} f_i)(x) = \bigvee_{i \in I}(f_i x)$.

Proof: By Theorem 5.9, $[X \to Y]$ is a d-space, and by Prop. 5.8, directed joins are pointwise. The empty join is the constant function $\lambda x.\, 0_Y$, and binary join is given by $f \vee g = \lambda x.\, fx \vee gx$. This is continuous since it is given by a λ-expression. $\square$

The class of d-spaces is closed under equalisers. This does not hold for join spaces, but at least we have:

Proposition 5.14 Retracts of join spaces are again join spaces.

Proof: Let $e : X \to Y$ and $r : Y \to X$ be continuous functions with $r \circ e = \mathsf{id}_X$. Assuming that Y is a join space, we must show that X is a join space. First, X is a d-space by Theorem 5.7 since it is (via e) the equaliser of $e \circ r : Y \to Y$ and id_Y. For all x in X, $0_Y \sqsubseteq ex$ holds, and thus $r0_Y \sqsubseteq r(ex) = x$; this gives the least element of X. Binary joins in X are given by $x_1 \vee x_2 = r(ex_1 \vee ex_2)$; this function is continuous since r, e, and join in Y are continuous. $\square$

6 d-Space Structures

Given a dcpo $D = (D, \sqsubseteq)$, there are in general several different convergence structures on the set D which define a d-space whose induced preorder is '$\sqsubseteq$'. These structures are called *d-space structures for* $(D, \sqsubseteq)$.

6.1 The Topological Structure

A topological structure on D with induced preorder '$\sqsubseteq$' is a d-space structure for D if and only if every open set is Scott open. Hence, the Scott topology defines the *strongest topological* d-space structure for D. This structure is denoted by '$\downarrow_{\mathrm{s}}$', and the resulting d-space $(D, \downarrow_{\mathrm{s}})$ by D_{s}. In a sloppy way, we call '$\downarrow_{\mathrm{s}}$' the *topological structure* of D.

If the given dcpo happens to be a complete lattice L, then L_{s} is a d-space with least element and binary join. Unfortunately, it is not always a join space, because $\vee : L_{\mathrm{s}} \times L_{\mathrm{s}} \to L_{\mathrm{s}}$ is not always continuous. For, the Compendium [5, Cor. II-1.12] contains a result that L_{s} is sober if '$\vee$' is continuous in L_{s}, but Isbell has found a complete lattice L where L_{s} is not sober [13].

6.2 The Strongest d-Space Structure

Now we look for the strongest d-space structure of all, which is strictly stronger than '$\downarrow_{\mathrm{s}}$' in general. A hint what this strongest structure might look like is given by the following fact:

Proposition 6.1 For every d-space X and filter $\mathcal{A}$ in X, $\mathrm{Lim}\,\mathcal{A} \supseteq \mathsf{cl}\,(\bigcup_{A \in \mathcal{A}} A^{\downarrow})$ holds, where 'cl' is closure in the Scott topology and $A^{\downarrow}$ is the set of lower bounds of A.

Proof: As a d-space, X is up-nice, and so $\mathrm{Lim}[A] = A^{\downarrow}$ holds for all $A \subseteq X$ by Prop. 4.1. For any A in $\mathcal{A}$, $\mathcal{A} \leq_{\mathrm{i}} [A]$, whence $A^{\downarrow} = \mathrm{Lim}[A] \subseteq \mathrm{Lim}\,\mathcal{A}$ by the subfilter axiom. Thus, $\bigcup_{A \in \mathcal{A}} A^{\downarrow} \subseteq \mathrm{Lim}\,\mathcal{A}$. Scott closure '$\mathsf{cl}$' can be added to the union since in a d-space all limit sets $\mathrm{Lim}\,\mathcal{A}$ are Scott closed. $\square$

The above proposition suggests that the strongest d-space structure is given by $\operatorname{Lim}\mathcal{A} = \mathsf{cl}\,(\bigcup_{A\in\mathcal{A}} A^{\downarrow})$. Indeed, this conjecture is true, and unlike the Scott topology, this definition even yields a join space if the given dcpo happens to be a complete lattice. These and other properties are shown in the sequel.

Definition 6.2 For every dcpo D, let '$\downarrow_{\mathrm{c}}$' be the convergence structure defined by $\mathcal{A} \downarrow_{\mathrm{c}} x$ iff $x \in \mathsf{cl}\,(\bigcup_{A\in\mathcal{A}} A^{\downarrow})$ where 'cl' is closure in the Scott topology and $A^{\downarrow}$ is the set of lower bounds of A. This structure is called the *cotopological structure* of D, and $D_{\mathrm{c}} = (D, \downarrow_{\mathrm{c}})$ is called a *cotopological dcpo.*

The term "cotopological" refers to the behaviour in the function space construction (see Theorem 7.11 and Cor. 7.21, or Section 7.6).

Let's prove that '$\downarrow_{\mathrm{c}}$' is the strongest d-space structure for D. First, we show that it is a convergence structure at all. If $\mathcal{A}' \leq_{\mathrm{i}} \mathcal{A}$, then $\mathcal{A} \subseteq \mathcal{A}'$, and thus $\mathsf{cl}\,(\bigcup_{A\in\mathcal{A}} A^{\downarrow}) \subseteq \mathsf{cl}\,(\bigcup_{A\in\mathcal{A}'} A^{\downarrow})$, which proves that the subfilter axiom is satisfied. The convergence $[x] \downarrow_{\mathrm{c}} x$ holds since $\mathsf{cl}\,(\bigcup_{A\in[x]} A^{\downarrow}) = \mathsf{cl}\,\{x\}^{\downarrow} = {\downarrow}x$.

Second, we show that the induced preorder '$\sqsubseteq_{\mathrm{c}}$' of D_{c} is the order '$\sqsubseteq$' of the given dcpo D. The calculation at the end of the previous paragraph shows $\operatorname{Lim}[x] = {\downarrow}x$. Hence, $y \sqsubseteq x$ implies $[x] \downarrow_{\mathrm{c}} y$, whence $y \sqsubseteq_{\mathrm{c}} x$. For the opposite implication, we note that the identity $D_{\mathrm{c}} \to D_{\mathrm{s}}$ is continuous by Prop. 6.1. Since continuous functions are monotonic, and '$\sqsubseteq$' is the specialisation preorder of D_{s}, $y \sqsubseteq_{\mathrm{c}} x$ implies $y \sqsubseteq x$.

Third, we show that D_{c} is a d-space. It is up-nice since $A^{\downarrow} = ({\uparrow}A)^{\downarrow}$, and hence, $\mathcal{A} \downarrow_{\mathrm{c}} x$ and ${\uparrow}\mathcal{A} \downarrow_{\mathrm{c}} x$ are equivalent. It is down-nice and a d-space structure since the limit sets $\operatorname{Lim}\mathcal{A}$ are Scott closed by definition. By Prop. 6.1, it is the strongest d-space structure for D.

We also show that the induced topology of D_{c} is the Scott topology. Since D_{c} is a d-space, every open set of D_{c} is Scott open by Cor. 5.4. By Prop. 6.1, the identity $D_{\mathrm{c}} \to D_{\mathrm{s}}$ is continuous, hence topologically continuous, and therefore, every Scott open set is open in D_{c}.

Finally, we note that every limit set $\operatorname{Lim}\mathcal{A}$ is Scott closed by definition, hence closed in the induced topology. This gives closure-niceness. Summarising, we have shown:

Theorem 6.3 For every dcpo D, '$\downarrow_{\mathrm{c}}$' is the strongest d-space structure for D. The space $D_{\mathrm{c}} = (D, \downarrow_{\mathrm{c}})$ is a closure-nice d-space, whose induced topology is the Scott topology of D.

We now present one of the simplest examples for $L_{\mathrm{c}} \neq L_{\mathrm{s}}$. Let L be the complete lattice which consists of a least element $\perp$, a greatest element $\top$, and two chains $a_1 \leq a_2 \leq \cdots$ and $b_1 \leq b_2 \leq \cdots$ which have the same join $\top$, but are otherwise unrelated. In L_{s}, the filter $\mathcal{F} = [{\uparrow}\{a_n, b_n\} \mid n \geq 1]$ converges to $\top$ (and to any other point as well) since every non-empty Scott open set contains a_n and b_n for some n. In L_{c} however, $\mathcal{F}$ does not converge to $\top$ since $\perp$ is the only lower bound of ${\uparrow}\{a_n, b_n\}$, and so, $\operatorname{Lim}_{\mathrm{c}}\mathcal{F} = \{\perp\}$.

The same example shows that cotopological dcpos are not always merge-nice. In L_{c}, the two filters $\mathcal{A} = [{\uparrow}a_n \mid n \geq 0]$ and $\mathcal{B} = [{\uparrow}b_n \mid n \geq 0]$ converge to $\top$ (direct from the definition, or from Lemma 5.1 (2)), but $\mathcal{A} \vee \mathcal{B} = \mathcal{F}$ does not converge to $\top$.

6.3 Alternative Characterisations of '$\downarrow_c$'

The definition of '$\downarrow_c$' in terms of Scott closure and lower bound operator $(-)^{\downarrow}$ can be rephrased in several equivalent ways:

Proposition 6.4 $x \in \mathsf{cl}\,(\bigcup_{A\in\mathcal{A}} A^{\downarrow})$
iff each Scott open neighbourhood O of x meets $A^{\downarrow}$ for some A in $\mathcal{A}$
iff for each Scott open neighbourhood O of x, there are $x' \in O$ and $A \in \mathcal{A}$ with $A \subseteq \uparrow x'$
iff for each Scott open neighbourhood O of x, there is $x' \in O$ such that $\uparrow x' \in \mathcal{A}$.

Here, the last formulation turns out to be the most useful in proofs. When we refer to Prop. 6.4, we always mean this last one.

The main weakness of Def. 6.2 and Prop. 6.4 is their reference to the Scott topology which is hard to characterise for arbitrary dcpos. Fortunately, there is a purely order-theoretic characterisation in case of complete lattices:

Proposition 6.5 In a complete lattice, $\mathcal{A} \downarrow_c x$ iff $x \leq \bigvee_{A\in\mathcal{A}} \bigwedge A$. This join is directed.

Proof: For every A in $\mathcal{A}$, $A^{\downarrow} = \downarrow \bigwedge A \subseteq \downarrow \bigvee_{A\in\mathcal{A}} \bigwedge A$ and so $\mathsf{cl}\,(\bigcup_{A\in\mathcal{A}} A^{\downarrow}) \subseteq \downarrow \bigvee_{A\in\mathcal{A}} \bigwedge A$. Conversely, if $x \leq \bigvee_{A\in\mathcal{A}} \bigwedge A$, then every Scott open neighbourhood of x contains $\bigwedge A$ for some A in $\mathcal{A}$. Since $\bigwedge A \in A^{\downarrow}$, Prop. 6.4 applies. $\square$

Thus, $\mathrm{Lim}\,\mathcal{A}$ has the form $\downarrow a$ where $a = \bigvee_{A\in\mathcal{A}} \bigwedge A$. This matches the third part of the defining theorem for join spaces (Theorem 5.11).

Corollary 6.6 If L is a complete lattice, then L_c is a join space; in particular, $\vee : L_c \times L_c \to L_c$ is continuous.

This property distinguishes L_c from L_s; for, $\vee : L_s \times L_s \to L_s$ is not always continuous (see Section 6.1 and the introduction).

For complete lattices, the order-theoretic convergence relation of Prop. 6.5 has been considered earlier. In the Compendium [5, II 1.1–1.8], the analogous relation for nets was taken as a motivation of the Scott topology which arises as the induced topology. In [17, 3], the convergence relation (for filters) was called "Scott convergence" (although it is *not* convergence in the Scott topology in general, cf. Theorem 7.3 below). In these papers, the "Scott convergence" was generalised from complete lattices to all posets in several different ways, which are all different from our definition of '$\downarrow_c$'.

7 Cotopological Dcpos

7.1 Basic Properties of Cotopological Dcpos

We have already seen that the induced topology of a cotopological dcpo is the Scott topology. A similar property holds for functions.

Theorem 7.1 Let D and E be dcpos and $f : D \to E$ a function. Then $f : D_c \to E_c$ is continuous, iff $f : D \to E$ is Scott continuous, iff $f : D_s \to E_s$ is continuous.

Proof: By Cor. 5.3, every continuous function between the d-spaces D_c and E_c is Scott continuous. Conversely, let $f : D \to E$ be Scott continuous, and $\mathcal{A} \downarrow_c x$ in D. If V is a Scott open neighbourhood of fx, then $f^- V$ is a Scott open neighbourhood of x, whence there is $x' \in f^- V$ with $\uparrow x' \in \mathcal{A}$. This gives $fx' \in V$ with $\uparrow fx' \supseteq f^+(\uparrow x') \in f^+\mathcal{A}$. Therefore $f^+\mathcal{A} \downarrow_c fx$ as required.

The second equivalence is well-known. $\square$

Corollary 7.2 $(-)_c$ and $(-)_s$ are full and faithful embeddings of DCPO into CONV.

Thus, D_c and D_s cannot be distinguished by the induced topology, nor by continuity of functions (of one argument, but recall that $\vee : L_c \times L_c \to L_c$ is always continuous, while $\vee : L_s \times L_s \to L_s$ is sometimes not continuous). The question of when D_c and D_s are identical is settled by the following equivalences:

Theorem 7.3 Let D be a dcpo. D_c is topological, iff $D_c = D_s$, iff D is continuous.

Proof: Since the induced topology of D_c is the Scott topology, D_c can only be topological if it equals D_s. Assume $D_c = D_s$. Then $\mathcal{N}(x) \downarrow_c x$, and so, for each Scott open $U \ni x$, there is $y \in U$ such that $\uparrow y \in \mathcal{N}(x)$, i.e., there is a Scott open V such that $x \in V \subseteq \uparrow y \subseteq U$. This "local supercompactness property" characterises continuous dcpos topologically. Conversely, if D is continuous, "local supercompactness" proves $\mathcal{N}(x) \downarrow_c x$, and so $\mathcal{A} \downarrow_s x \Rightarrow \mathcal{A} \leq_i \mathcal{N}(x)$ $\Rightarrow \mathcal{A} \downarrow_c x$, whence $D_c = D_s$. $\square$

We already know that all cotopological dcpos D_c are up-nice, down-nice, and closure-nice. Now we consider merge-niceness in the case of dcpos with binary meets.

Theorem 7.4 Let D be a dcpo with binary meets '$\sqcap$'. Then D_c is merge-nice iff $\sqcap : D \times D \to D$ is Scott continuous.

Proof: If meet is Scott continuous, then $m_a = \lambda b.\, a \sqcap b : D \to D$ is Scott continuous for every a in D. Assume $\mathcal{A} \downarrow_c x$ and $\mathcal{B} \downarrow_c x$. To prove $\mathcal{A} \cap \mathcal{B} \downarrow_c x$, we apply Prop. 6.4. Thus, let x be in a Scott open set O. By Scott continuity of m_x, $U = m_x^- O$ is Scott open as well, and contains x since $x \sqcap x = x \in O$. Because of $\mathcal{A} \downarrow_c x \in U$, there is a in U with $\uparrow a \in \mathcal{A}$. Since a is in $U = m_x^- O$, $a \sqcap x$ is in O, and therefore, $V = m_a^- O$ is another Scott open neighbourhood of x. Because of $\mathcal{B} \downarrow_c x \in V$, there is b in V with $\uparrow b \in \mathcal{B}$. Then $c = a \sqcap b$ is in O, and $\uparrow c$ as a superset of both $\uparrow a$ and $\uparrow b$ is in $\mathcal{A} \cap \mathcal{B}$. This concludes the proof of $\mathcal{A} \cap \mathcal{B} \downarrow_c x$.

Conversely, assume merge-niceness, and consider a in D and a directed set Δ. Let $b = a \sqcap \bigsqcup \Delta$ and $c = \bigsqcup_{d \in \Delta}(a \sqcap d)$. The relation $b \sqsupseteq c$ always holds. We have $[a] \downarrow_c b$ since $b \sqsubseteq a$, and $\langle \Delta \rangle \downarrow_c b$ by $b \sqsubseteq \bigsqcup \Delta$ and Lemma 5.1. By merge-niceness, $[a] \cap \langle \Delta \rangle \downarrow_c b$ follows. The sets C in $[a] \cap \langle \Delta \rangle$ contain a and $\uparrow d$ for some d in Δ. Thus, $C^{\downarrow} \subseteq \downarrow(a \sqcap d) \subseteq \downarrow c$ holds, whence $\mathsf{cl}\,(\bigcup_{C \in [a] \cap \langle \Delta \rangle} C^{\downarrow}) \subseteq \downarrow c$ follows, and so b, as a limit point of $[a] \cap \langle \Delta \rangle$, is in $\downarrow c$ as well. $\square$

While the above theorem is kind of bad news concerning the niceness of cotopological lattices, it gives at least a new proof of an old theorem: in a continuous dcpo with binary meets, the cotopological structure is merge-nice because it coincides with the topological structure, and therefore, meet is Scott continuous.

7.2 Products of Cotopological Dcpos

Given a family $(D_i)_{i \in I}$ of dcpos, we want to compare $(\prod_{i \in I} D_i)_\mathrm{c}$ and $\prod_{i \in I}(D_i)_\mathrm{c}$.

Proposition 7.5 The identity function $(\prod_{i \in I} D_i)_\mathrm{c} \to \prod_{i \in I}(D_i)_\mathrm{c}$ is continuous.

Proof: The projections $\prod_{i \in I} D_i \to D_i$ are Scott continuous, hence continuous $(\prod_{i \in I} D_i)_\mathrm{c} \to (D_i)_\mathrm{c}$. □

For complete lattices, the opposite direction is easily obtained:

Theorem 7.6 For any family $(L_i)_{i \in I}$ of complete lattices, $(\prod_{i \in I} L_i)_\mathrm{c} = \prod_{i \in I}(L_i)_\mathrm{c}$ holds.

Proof: If $\mathcal{A} \downarrow x$ in $\prod_{i \in I}(L_i)_\mathrm{c}$, then $\pi_i^+ \mathcal{A} \downarrow_\mathrm{c} x_i$, i.e., $x_i \le \bigvee_{A \in \mathcal{A}} \bigwedge \pi_i^+ A$ for all i in I. Since projections preserve all joins and meets, we get $x_i \le \pi_i(\bigvee_{A \in \mathcal{A}} \bigwedge A)$ for all i in I, whence $\mathcal{A} \downarrow_\mathrm{c} x$. □

On the positive side, we have in particular $(L \times L)_\mathrm{c} = L_\mathrm{c} \times L_\mathrm{c}$. This shows once again that $\vee : L_\mathrm{c} \times L_\mathrm{c} \to L_\mathrm{c}$ is continuous (because it is Scott continuous). On the other hand, $\vee : L_\mathrm{s} \times L_\mathrm{s} \to L_\mathrm{s}$ is not always continuous. This gives an example where the induced topology of the product is not the product of the induced topologies. For, the induced topology of $L_\mathrm{c} \times L_\mathrm{c} = (L \times L)_\mathrm{c}$ is the Scott topology, while L_c with the induced topology is L_s, and if the product topology of $L_\mathrm{s} \times L_\mathrm{s}$ were the Scott topology as well, then $\vee : L_\mathrm{s} \times L_\mathrm{s} \to L_\mathrm{s}$ would be Scott continuous.

Theorem 7.6 cannot be fully generalised to dcpos. Consider for instance the family $(D_i)_{i \in I}$ where I is infinite and all D_i are equal to the discrete two-point dcpo D. Then $(D_\mathrm{c})^I = (D_\mathrm{s})^I$ since D is algebraic, and thus $(D_\mathrm{c})^I$ is topological with a non-discrete topology. Yet the induced topology of $(D^I)_\mathrm{c}$ is the Scott topology, which is discrete. Therefore, $(D^I)_\mathrm{c} \ne (D_\mathrm{c})^I$.

Finite products are okay:

Theorem 7.7 For two dcpos D and E, $(D \times E)_\mathrm{c} = D_\mathrm{c} \times E_\mathrm{c}$ holds.

Proof: Let $\mathcal{C} \downarrow (a, b)$ in $D_\mathrm{c} \times E_\mathrm{c}$, i.e., $\mathcal{A} \downarrow_\mathrm{c} a$ and $\mathcal{B} \downarrow_\mathrm{c} b$ where $\mathcal{A} = \pi_1^+ \mathcal{C}$ and $\mathcal{B} = \pi_2^+ \mathcal{C}$. We have to show $\mathcal{C} \downarrow_\mathrm{c} (a, b)$, so let W be a Scott open neighbourhood of (a, b). Note that W is not necessarily open in the product topology; therefore the "usual" way to proceed is not possible.

Let $U = \{x \in D \mid (x, b) \in W\}$. This is a Scott open neighbourhood of a since the function $\lambda x. (x, b)$ is Scott continuous. Because of $\mathcal{A} \downarrow_\mathrm{c} a$, there is a' in U with $\uparrow a' \in \mathcal{A}$, or $\mathcal{A} \le_\mathrm{i} [\uparrow a']$. Since a' is in U, (a', b) is in W. Now, we do the same the other way round: let $V = \{y \in E \mid (a', y) \in W\}$. By $\mathcal{B} \downarrow_\mathrm{c} b$, there is b' in V with $\mathcal{B} \le_\mathrm{i} [\uparrow b']$. Then we have $(a', b') \in W$ and $\mathcal{C} \le_\mathrm{i} \mathcal{A} \times \mathcal{B} \le_\mathrm{i} [\uparrow a'] \times [\uparrow b'] = [\uparrow (a', b')]$, i.e., $\uparrow (a', b') \in \mathcal{C}$. □

Even infinite products are okay if almost all dcpos are pointed (which was not true in the counterexample above). This result subsumes Theorem 7.6, but the proof is much more involved.

Theorem 7.8 Let $(D_i)_{i \in I}$ be a family of dcpos with the property that almost all D_i have a least element $\bot_i$. Then $(\prod_{i \in I} D_i)_\mathrm{c} = \prod_{i \in I}(D_i)_\mathrm{c}$ holds.

Proof: We have to show that the identity function $\mathsf{id} : \prod_{i\in I}(D_i)_{\mathrm{c}} \to (\prod_{i\in I} D_i)_{\mathrm{c}}$ is continuous. Let $B \subseteq_{\mathrm{fin}} I$ be the set of indices of the non-pointed dcpos. For every finite subset J of I with $J \supseteq B$, the projection function $p_J : \prod_{i\in I}(D_i)_{\mathrm{c}} \to \prod_{j\in J}(D_j)_{\mathrm{c}}$ is continuous. By Theorem 7.7, $\prod_{j\in J}(D_j)_{\mathrm{c}}$ is the same as $(\prod_{j\in J} D_j)_{\mathrm{c}}$. Let $e_J : \prod_{j\in J} D_j \to \prod_{i\in I} D_i$ be the function defined by

$$(e_J x)_i \;=\; \begin{cases} x_i & \text{if } i \in J \\ \perp_i & \text{otherwise.} \end{cases}$$

This function is Scott continuous, and hence continuous $(\prod_{j\in J} D_j)_{\mathrm{c}} \to (\prod_{i\in I} D_i)_c$ by Theorem 7.1. Together, we have a continuous function $f_J = e_J \circ p_J : \prod_{i\in I}(D_i)_{\mathrm{c}} \to (\prod_{i\in I} D_i)_{\mathrm{c}}$, which leaves the components in J unchanged and maps all other components x_i to $\perp_i$. The family $(f_J)_J$ where J ranges over the finite subsets of I that contain B is a directed family of continuous functions with join id. By Prop. 5.8, id is continuous. □

Again, this gives a new proof of an old theorem: if $(D_i)_{i\in I}$ is a family of continuous dcpos where almost all are pointed, then $(\prod_{i\in I} D_i)_{\mathrm{c}} = \prod_{i\in I}(D_i)_{\mathrm{c}} = \prod_{i\in I}(D_i)_{\mathrm{s}}$ is topological, and thus $\prod_{i\in I} D_i$ is continuous again.

7.3 Function Spaces from Topological to Cotopological

Now, we consider the situation where X is topological and $Y = D_{\mathrm{c}}$ is a cotopological dcpo. From Theorem 5.9, we know that $[X \to D_{\mathrm{c}}]$ is a d-space. Hence, the continuous functions from X to D_{c} form a dcpo $(X \to D_{\mathrm{c}})$, and the identity $(X \to D_{\mathrm{c}})_{\mathrm{c}} \to [X \to D_{\mathrm{c}}]$ is continuous since '$\downarrow_{\mathrm{c}}$' is the strongest d-space structure for $(X \to D_{\mathrm{c}})$.

For the opposite direction, one cannot hope for much. We have already seen a counterexample in Section 7.2 where X is an infinite discrete space and D is the discrete two-point dcpo. The product experience suggests to require D to be pointed. But even this is not enough, since any positive result would imply a similar result for continuous dcpos (the exact argument will be presented in Section 7.6), but it is well-known that the function space of two pointed continuous dcpos is not continuous in general.

However, we are able to show a result for complete lattices L. Before we come to this, we consider how continuous functions $X \to D_{\mathrm{c}}$ are characterised. As all CONV-continuous functions, they are also TOP-continuous, i.e., each continuous function $X \to D_{\mathrm{c}}$ is also continuous $X \to D_{\mathrm{s}}$. The converse does not hold in general; consider for instance the identity $D_{\mathrm{s}} \to D_{\mathrm{s}}$ in case $D_{\mathrm{c}} \neq D_{\mathrm{s}}$.

A function $f : X \to D_{\mathrm{c}}$ is continuous, iff $\mathcal{A} \downarrow x$ implies $f^+\mathcal{A} \downarrow_{\mathrm{c}} fx$, iff $f^+\mathcal{N}(x) \downarrow_{\mathrm{c}} fx$ for all x in X. The latter means that for every Scott open neighbourhood V of fx, there are y in V and an open neighbourhood U of x such that $f^+U \subseteq {\uparrow}y \subseteq V$. (Topological continuity would be similar, but without '${\uparrow}y$' in between.)

If D is a complete lattice, the condition $f^+\mathcal{N}(x) \downarrow_{\mathrm{c}} fx$ for all x in X means $fx \leq \bigvee_{U\in\mathcal{N}(x)} \bigwedge f^+U$ by Prop. 6.5. Here, '$\leq$' may be replaced by '$=$' since '$\geq$' always holds. Summarising, we have:

Proposition 7.9

(1) Let X be a topological space and D a dcpo. A function $f : X \to D_c$ is continuous iff for every Scott open neighbourhood V of fx, there are y in V and an open neighbourhood U of x such that $f^+U \subseteq \uparrow y \subseteq V$.

(2) If D is moreover a complete lattice, this is equivalent to $fx = \bigvee_{U \in \mathcal{N}(x)} \bigwedge f^+U$ for all x in X (the relation that matters is '$\leq$').

Proposition 7.10 Let X be a topological space and L a complete lattice. Then $[X \to L_c]$ is a complete lattice again where joins are pointwise and meets are given by $(\bigwedge F)(x) = \bigvee_{U \in \mathcal{N}(x)} \bigwedge (F \cdot U)$.

Proof: By Theorem 5.13, $[X \to L_c]$ is a join space, and joins are given pointwise. For meets, let $g = (x \mapsto \bigvee_{U \in \mathcal{N}(x)} \bigwedge (F \cdot U))$. First, g is continuous by Prop. 7.9 (1) since for Scott open $V \ni gx$, there is U in $\mathcal{N}(x)$ such that $\bigwedge (F \cdot U) \in V$. For each u in U, $\bigwedge (F \cdot U) \leq gu$ holds, whence $g^+U \subseteq \uparrow \bigwedge (F \cdot U)$. Second, g is a lower bound of F since for all f in F, x in X and $U \in \mathcal{N}(x)$, $\bigwedge (F \cdot U) \leq fx$, whence $gx \leq fx$. Finally, g is the greatest lower bound since for all continuous lower bounds h of F, Prop. 7.9 (2) gives $hx = \bigvee_{U \in \mathcal{N}(x)} \bigwedge h^+U \leq \bigvee_{U \in \mathcal{N}(x)} \bigwedge (F \cdot U) = gx$. □

Using these results, we may now show:

Theorem 7.11 If X is topological and L a complete lattice, then $[X \to L_c]$ is a complete lattice again, and the function space structure coincides with the cotopological structure.

Proof: We have to show that $\mathcal{F} \downarrow g$ implies $\mathcal{F} \downarrow_c g$, which means $g \leq \bigvee_{F \in \mathcal{F}} \bigwedge F$. For each x in X, we have $\mathcal{N}(x) \downarrow x$, and thus $\mathcal{F} \cdot \mathcal{N}(x) \downarrow_c gx$, i.e., $gx \leq \bigvee_{F \in \mathcal{F}} \bigvee_{U \in \mathcal{N}(x)} \bigwedge (F \cdot U)$. By the characterisation of meets, the latter equals $\bigvee_{F \in \mathcal{F}} (\bigwedge F)(x)$. Since join is pointwise, this is $(\bigvee_{F \in \mathcal{F}} \bigwedge F)(x)$. This shows $g \leq \bigvee_{F \in \mathcal{F}} \bigwedge F$ as required. □

One may try to extend this result from complete lattices to a more general class of dcpos. Bounded-complete dcpos are good candidates, and one may consider analogues of L-domains or SFP domains.

7.4 An Injectivity Result

In a category, an object Z is *injective* for an arrow $f : X \to Y$ if for every arrow $g : X \to Z$, there is some (not necessarily unique) 'extension' $h : Y \to Z$ such that $h \circ f = g$.

We specialise this general notion for our purposes: For a subclass C of convergence spaces, let's say a convergence space Z is *C-injective* if it is injective for all pre-embeddings $e : X \to Y$ between objects X and Y from C.

A topological space is TOP-injective if and only if it is a continuous lattice with the Scott topology (this is a slight modification of the results in the Compendium [5, Section II-3]). In contrast, we have the following result:

Theorem 7.12 Every cotopological lattice is TOP-injective: if X and Y are topological spaces, $e : X \to Y$ is a pre-embedding, and L a complete lattice, then for every continuous function $f : X \to L_c$, there is a continuous 'extension' $g : Y \to L_c$ such that $g \circ e = f$. It is explicitly given by $gy = \bigvee_{V \in \mathcal{N}(y)} \bigwedge f^+(e^- V)$, and it is the greatest among the continuous functions h satisfying $h \circ e \leq f$.

Proof: First, we show that g is continuous using Prop. 7.9 (2). Thus, we need to show

$$\bigvee_{V \in \mathcal{N}(y)} \bigwedge f^+(e^- V) \leq \bigvee_{V \in \mathcal{N}(y)} \bigwedge g^+ V.$$

For any open neighbourhood V of y and any v in V, $\bigwedge f^+(e^- V) \leq gv$ holds by definition of g, whence $\bigwedge f^+(e^- V) \leq \bigwedge g^+ V$.

Second, we show $g(ex) = fx$ for all x in X. Using the definition of g and expanding fx with Prop. 7.9 (2), the equation becomes

$$\bigvee_{V \in \mathcal{N}(ex)} \bigwedge f^+(e^- V) = \bigvee_{U \in \mathcal{N}(x)} \bigwedge f^+ U.$$

For every open neighbourhood V of ex, $U = e^- V$ is an open neighbourhood of x by continuity of e. Since e is a pre-embedding, each open neighbourhood U of x can be written as $U = e^- V$ for some open V of Y, which obviously is a neighbourhood of ex. These arguments prove the above equality.

Third, we show that $h \circ e \leq f$ implies $h \leq g$. Expanding hy with Prop. 7.9 (2) and using the definition of g, the relation $hy \leq gy$ becomes

$$\bigvee_{V \in \mathcal{N}(y)} \bigwedge h^+ V \leq \bigvee_{V \in \mathcal{N}(y)} \bigwedge f^+(e^- V).$$

To prove this, it suffices to show $f^+(e^- V) \subseteq \uparrow h^+ V$ for all open neighbourhoods V of y. This inclusion holds since for all x in $e^- V$, ex is in V, and thus $fx \geq h(ex) \in h^+ V$. □

This theorem generalises the fact that continuous lattices are TOP-injective. It shows that in the larger category CONV, there are non-continuous lattices which are TOP-injective; indeed, any complete lattice whatsoever can be made TOP-injective by imposing the cotopological structure '$\downarrow_c$' on it. For the moment, we are not able to show that cotopological lattices are the only TOP-injective spaces.

The theorem breaks down without the condition that X and Y are topological. If pre-embeddings between arbitrary convergence spaces are taken into account, then not even Ω is injective; recall Example 3.1 of a convergence space Y with a subspace X that has more opens than the ones coming from the subspace topology.

7.5 Topological Function Spaces

If D is a dcpo and Y a topological space, the continuous functions $D_c \to Y$ are topologically characterised, and therefore coincide with the continuous functions $D_s \to Y$ (yet $[D_c \to Y]$ and $[D_s \to Y]$ have different convergence structures in general).

Our goal in this section is to prove that the function space $[D_c \to Y]$ is topological, and its topology is the "point-open" topology, i.e., the topology with subbasic opens $\langle x \to V \rangle = \{f \in [D_c \to Y] \mid fx \in V\}$ where x ranges over the elements of D and V over the opens of Y. Actually, we shall prove results that are more general than this, providing a full characterisation of when $[X \to Y]$ is topological. The statement about cotopological dcpos will be derived at the end. We start out with some general remarks on function spaces.

Proposition 7.13 If X is empty, then $[X \to Y] \cong \mathbf{1}$ is always topological. If X is not empty, then $[X \to Y]$ is topological only if Y is topological.

Proof: $[\emptyset \to Y]$ has only one element, and all convergence spaces with one element are isomorphic to the terminal topological space $\mathbf{1}$. If there is some x_0 in X, then Y is a retract of $[X \to Y]$ by means of $\lambda y.\, \lambda x.\, y : Y \to [X \to Y]$ and $\lambda f.\, f x_0 : [X \to Y] \to Y$. Hence, Y is a subspace of $[X \to Y]$, and thus Y is topological if $[X \to Y]$ is topological. □

Because of the above proposition, we can concentrate on the case that Y is topological. We shall see that the function space $[X \to \Omega]$ plays a special role. Since continuous functions from X to Sierpinski space Ω correspond to open sets of X, we introduce the alternative notation ΩX for $[X \to \Omega]$. The points of ΩX can be considered as open sets or as continuous functions to Ω. Set view and function view are linked by $Ox = 1 \iff x \in O$.

Proposition 7.14 If Y is topological, then the function space structure of $[X \to Y]$ is the initial structure for the functions $\lambda f.\, f^- V : [X \to Y] \to \Omega X$ where V ranges over some subbasis of Y.

Proof: Let $\mathcal{S}$ be a subbasis of the topology of Y. The function $e : Y \to \prod_{V \in \mathcal{S}} \Omega$ with $(ey)_V = Vy$ is a (topological) pre-embedding. Hence, $e^X : [X \to Y] \to [X \to \prod_{V \in \mathcal{S}} \Omega]$ is a pre-embedding as well (see Section 3.3). Now, $[X \to \prod_{V \in \mathcal{S}} \Omega] \cong \prod_{V \in \mathcal{S}} [X \to \Omega] \cong \prod_{V \in \mathcal{S}} \Omega X$ holds. Hence, we obtain a pre-embedding $E : [X \to Y] \to \prod_{V \in \mathcal{S}} \Omega X$, and thus, $[X \to Y]$ carries the initial structure for the family $(\pi_V \circ E)_{V \in \mathcal{S}}$. Now, let's see what these functions actually do:

$$\pi_V(Ef)(x) \;=\; \pi_V(e^X f\, x) \;=\; \pi_V(e(fx)) \;=\; V(fx) \;=\; (f^- V)(x)$$

so $\pi_V \circ E = \lambda f.\, f^- V$ as claimed. □

Theorem 7.15 If Y and ΩX are topological, then $[X \to Y]$ is topological. In this case, a subbasis of the topology of $[X \to Y]$ is given by the sets $\langle \mathcal{U} \leftarrow V \rangle = \{f \in [X \to Y] \mid f^- V \in \mathcal{U}\}$, where $\mathcal{U}$ ranges over a subbasis of ΩX, and V over a subbasis of Y.

Proof: The property to be topological is preserved by initial constructions. Hence, $[X \to Y]$ is topological by Prop. 7.14. A subbasis of this initial topology is given by the sets $(\lambda f.\, f^- V)^- \mathcal{U}$, where V ranges over a subbasis of Y and $\mathcal{U}$ over a subbasis of ΩX. The observation $(\lambda f.\, f^- V)^- \mathcal{U} \;=\; \langle \mathcal{U} \leftarrow V \rangle$ concludes the proof. □

Since $\Omega X \cong [X \to \Omega]$ is a special case of $[X \to Y]$, we may conclude:

Corollary 7.16 For a convergence space X, the following are equivalent:

(1) ΩX is topological.

(2) For all topological spaces Y, $[X \to Y]$ is topological.

If X is restricted to be a topological space, then $\Omega X \cong [X \to \Omega]$ is a cotopological lattice by Theorem 7.11. By Theorem 7.3, a cotopological lattice is topological if and only if it is a continuous lattice; in this case it will carry the Scott topology. This gives the following corollary which was already known [15, Theorem 2.16].

Corollary 7.17 For a topological space X, the following are equivalent:

(1) ΩX is a continuous lattice.

(2) For all topological spaces Y, $[X \to Y]$ is topological.

In this case, the topology of $[X \to Y]$ is the *Isbell topology*: it has a subbasis consisting of the sets $\langle \mathcal{U} \leftarrow V \rangle$ where $\mathcal{U}$ ranges over the Scott open sets of ΩX and V over the open sets of Y.

If ΩX is a continuous lattice, every Scott open set of ΩX is a union of Scott open filters, which correspond to the compact upper sets of the soberification of X. Thus, the Isbell topology in Cor. 7.17 can be replaced by the compact-open topology if X is sober.

It is remarkable that the above results could be obtained without actually looking into the convergence structure of $\Omega X \cong [X \to \Omega]$. This is done now since it is needed for the results to follow.

Proposition 7.18 Let X be any convergence space. In ΩX, $\mathcal{F} \downarrow U$ holds iff for all $x \in U$ and all $\mathcal{A} \downarrow_X x$, there is $\mathcal{U} \in \mathcal{F}$ with $\bigcap \mathcal{U} \in \mathcal{A}$.[1]

Proof: By definition of the function space structure, $\mathcal{F} \downarrow U$ holds iff $\mathcal{A} \downarrow x$ implies $\mathcal{F} \cdot \mathcal{A} \downarrow Ux$. This refers to the convergence structure of Ω, where all filters converge to 0. Thus, we may restrict to the case $Ux = 1$, i.e., $x \in U$, and note that $\mathcal{F} \cdot \mathcal{A} \downarrow 1$ iff $\{1\} \in \mathcal{F} \cdot \mathcal{A}$, iff there are $\mathcal{U} \in \mathcal{F}$ and $A \in \mathcal{A}$ such that $\mathcal{U} \cdot A \subseteq \{1\}$. The latter means $a \in O$ for all $a \in A$ and $O \in \mathcal{U}$, or $A \subseteq O$ for all $O \in \mathcal{U}$, or $A \subseteq \bigcap \mathcal{U}$. Finally, the existence of A in $\mathcal{A}$ with $A \subseteq \bigcap \mathcal{U}$ is equivalent to $\bigcap \mathcal{U} \in \mathcal{A}$. $\Box$

With this knowledge about the convergence structure of ΩX, we can derive a (clumsy) criterion for ΩX to be topological.

Proposition 7.19 For a convergence space X and a set $\mathcal{B}$ of subsets of ΩX, the following are equivalent:

(1) The space of open sets ΩX is topological with basis $\mathcal{B}$.

(2) All elements of $\mathcal{B}$ are open in the induced topology of ΩX, and for all $\mathcal{A} \downarrow_X x$ and induced open neighbourhoods U of x, there is a set $\mathcal{U} \in \mathcal{B}$ (a set of open sets) with $U \in \mathcal{U}$ and $\bigcap \mathcal{U} \in \mathcal{A}$.

Proof: If ΩX is topological with basis $\mathcal{B}$, then all elements of $\mathcal{B}$ are induced open since the induced topology of ΩX is the original topology. Consider the situation $\mathcal{A} \downarrow_X x \in U$ for some open U. Since ΩX is topological, $\mathcal{N}(U) \downarrow U$ holds. By Prop. 7.18, there is some $\mathcal{V}$ in $\mathcal{N}(U)$ with $\bigcap \mathcal{V} \in \mathcal{A}$. Since $\mathcal{B}$ is a basis of the topology of ΩX, there is some $\mathcal{U} \in \mathcal{B}$ with $U \in \mathcal{U} \subseteq \mathcal{V}$. Then $\bigcap \mathcal{U} \supseteq \bigcap \mathcal{V}$, and thus, $\bigcap \mathcal{U}$ is in $\mathcal{A}$ as well.

[1] $\mathcal{F}$ is a filter in ΩX, i.e., a set of sets of open sets, $\mathcal{U}$ is a set of open sets, $\bigcap \mathcal{U}$ a set, and $\mathcal{A}$ a set of sets.

For the opposite direction, we need to show that the convergence structure '$\downarrow$' of ΩX satisfies $\mathcal{F} \downarrow U$ iff $\mathcal{F} \leq_{\mathrm{i}} \mathcal{N}(U)$ where $\mathcal{N}(U) = [\mathcal{U} \in \mathcal{B} \mid U \in \mathcal{U}]$ is the neighbourhood filter of the topology generated by $\mathcal{B}$. First, $\mathcal{F} \downarrow U$ implies $\mathcal{F} \leq_{\mathrm{i}} \mathcal{N}(U)$ since the sets $\mathcal{U} \in \mathcal{B}$ are open by hypothesis. For the opposite implication, it suffices to show $\mathcal{N}(U) \downarrow U$. We use Prop. 7.18 for this purpose. So assume $\mathcal{A} \downarrow x \in U$. By hypothesis, there is $\mathcal{U} \in \mathcal{B}$ with $U \in \mathcal{U}$ (whence $\mathcal{U} \in \mathcal{N}(U)$) and $\bigcap \mathcal{U} \in \mathcal{A}$. $\square$

We are now interested in the special case where ΩX is topological with the *point topology*, i.e., the topology with subbasis $\mathcal{O}(x) = \{U \in \Omega X \mid x \in U\}$ where x ranges over the points of X. A basis of the point topology is given by the sets $\mathcal{O}(F) = \{U \in \Omega X \mid F \subseteq U\}$ where F ranges over the finite subsets of X.

Theorem 7.20 For a convergence space X, the following are equivalent:

(1) ΩX is topological with the point topology.

(2) For all topological spaces Y, $[X \to Y]$ is topological with the point-open topology.

(3) X is *locally finitary*, i.e., for all $\mathcal{A} \downarrow_X x$ and induced open neighbourhoods U of x, there is a finite subset $F \subseteq U$ with $\uparrow F \in \mathcal{A}$.

Proof: For the implication (2) $\Rightarrow$ (1) choose $Y = \Omega$ and note that $\langle x \to \{1\}\rangle = \mathcal{O}(x)$. The implication (1) $\Rightarrow$ (2) is a special instance of Theorem 7.15; note that $\langle \mathcal{O}(x) \leftarrow V\rangle = \langle x \to V\rangle$. The equivalence (1) $\Leftrightarrow$ (3) is Prop. 7.19; note that $U \in \mathcal{O}(F)$ iff $F \subseteq U$, and $\bigcap \mathcal{O}(F) = \uparrow F$. The extra condition in Prop. 7.19 that the basic sets $\mathcal{O}(F)$ are open in the induced topology of ΩX does not occur here since these sets are always induced open. For, $\mathcal{O}(F) = \bigcap_{x \in F} \mathcal{O}(x)$, and $\mathcal{O}(x) = (\lambda O.\, Ox)^{-}\{1\}$, where $\lambda O.\, Ox : \Omega X \to \Omega$ is continuous. $\square$

By Prop. 6.4, cotopological dcpos are locally finitary with a singleton set F. Therefore, we have

Corollary 7.21 If D is a dcpo and Y a topological space, then $[D_{\mathrm{c}} \to Y]$ is again topological, and its topology is the point-open topology.

7.6 Summary

With respect to function spaces, we have shown the following properties:

(1) If X is a cotopological dcpo and Y a topological space, then $[X \to Y]$ is a topological space (with the point-open topology) (Cor. 7.21).

(2) If X is a topological space and Y a cotopological lattice, then $[X \to Y]$ is again a cotopological lattice (Theorem 7.11).

These properties are the reason for the name "cotopological".

Statement (2) cannot be extended to cotopological pointed dcpos: Consider two continuous pointed dcpos D and E. Continuous dcpos are both cotopological and topological, and so $[D \to E]$ is topological by (1). If statement (2) were applicable, then $[D \to E]$ would be cotopological as well, and hence continuous, but we know that the function space of continuous pointed dcpos is not always continuous.

If the two statements are applied to the case $Y = \Omega$ which is both topological and cotopological, then we obtain:

(1) X cotopological $\Rightarrow$ ΩX topological $\Rightarrow$ $\Omega^2 X$ cotopological;

(2) X topological $\Rightarrow$ ΩX cotopological $\Rightarrow$ $\Omega^2 X$ topological.

Here, $\Omega^2 X$ is an abbreviation for $\Omega(\Omega X) = [[X \to \Omega] \to \Omega]$. The construction $X \mapsto \Omega X$ is the object part of a contravariant functor Ω with $\Omega f = f^-$, and so Ω^2 is a (covariant) functor in CONV. Statement (2) shows that this functor cuts down to an endofunctor of TOP. It can be described in purely topological terms as follows: for a topological space X, the points of $\Omega^2 X$ are Scott open sets of open sets, and the topology of $\Omega^2 X$ has subbasis $\mathcal{O}(U) = \{\mathcal{U} \in \Omega^2 X \mid U \in \mathcal{U}\}$ where U ranges over the opens of X.

Considering $\Omega^2 X$ as $[\Omega X \to \Omega]$, we may restrict to functions preserving finite joins and call the result LX. The elements of LX are in one-to-one correspondence with the closed sets C of X; this works for all convergence spaces X. Since subspaces of topological spaces are again topological, we see that L restricts to an endofunctor in TOP. In this case, the topology of LX has subbasis $\Diamond U = \{C \in LX \mid C \cap U \neq \emptyset\}$, i.e., we have obtained the familiar lower power space construction.

We may also restrict the functions in $[\Omega X \to \Omega]$ to those which preserve finite meets and call the result UX. Again, we see that U restricts to an endofunctor in TOP. The elements of UX are then Scott open filters of open sets, which are in one-to-one correspondence with compact upper sets K of X if X is sober. In this case, the topology of UX has basis $\Box U = \{K \in UX \mid K \subseteq U\}$, i.e., we have obtained the familiar upper power space construction.

Let R be the continuous lattice $[0, \infty]$. For any convergence space X, let VX be the subspace of $[\Omega X \to R]$ which consists of all strict and modular functions ($\nu\emptyset = 0$ and $\nu(U \cap V) + \nu(U \cup V) = \nu U + \nu V$). Again, V cuts down to an endofunctor in TOP. In this case, continuity of $\nu : \Omega X \to R$ means Scott continuity, and the topology of VX is the point-open topology, i.e., we have exactly obtained the ad-hoc definition of the "space of valuations" in [6].

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