Title: Similarity, Topology, and Uniformity
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Extended Abstract: Metric spaces have been generalized in many different ways: symmetry has been dropped, self-distances need not be 0, and the target domain of the distance function has been generalized from $\mathbb{R}^+$ to more general domains. We generalize these generalizations even further to a state without axioms and with arbitrary topological $\mathcal{T}_0$ spaces S as possible target domains. For convenience, we order these target domains by their specialization relation, which corresponds to the opposite of the usual ordering in case of $\mathbb{R}^+$. Therefore, we speak of similarity instead of distance (if a point x moves toward a point y, the distance between x and y shrinks, but their similarity increases).

Definitions: A *generalized similarity space* or shortly *gss* is a tuple (X, S, σ) where X is a set (the set of points), S is a $\mathcal{T}_0$ topological space, and $\sigma : X \times X \to S$ is a function (the similarity function). A gss is *symmetric* if $\sigma(x, x') = \sigma(x', x)$ holds for all $x, x' \in X$. Let x be a point of X and u an open set of S. The *right and left pre-open balls* about x with radius u are defined as $\mathrm{B}^{\mathrm{R}}(x, u) = \{x' \in X \mid \sigma(x, x') \in u\}$ and $\mathrm{B}^{\mathrm{L}}(x, u) = \{x' \in X \mid \sigma(x', x) \in u\}$.

This covers pseudo-quasi-metric spaces and partial metric spaces by taking $S = \mathbb{R}^+$ with the Scott topology of the opposite ordering, i.e., $u = [0, r)$.

We first study the properties of a single gss to find out which hypotheses are needed to prove results known from more familiar classes of generalized metrics. For instance, to show that the pre-open balls are open in the induced topology, a weak form of triangle inequality is sufficient in which the binary operation that takes over the role of addition is not required to be commutative or associative and may even vary dependent on the "middle point" of the triangle.

Definition: A gss (X, S, σ) is *locally transitive* if there is a family $(*_y)_{y\in X}$ of functions $*_y : S \times S \to S$ continuous in each argument separately such that

$$\sigma(x, y) *_y \sigma(y, z) \le \sigma(x, z), \quad \sigma(x, y) *_y \sigma(y, y) = \sigma(x, y), \quad \sigma(y, y) *_y \sigma(y, z) = \sigma(y, z).$$

It is *globally transitive* if it is locally transitive and the operations $*_y$ do not depend on y.

Proposition: The left and right pre-open balls are open in locally transitive gss.

Note that this covers partial metrics: After turning around $\mathbb{R}^+$ and its ordering, the triangle inequality for partial metrics becomes $p(x, z) \ge p(x, y) + p(y, z) - p(y, y)$. With the definition $a *_y b = a + b - p(y, y)$, the resulting gss is locally transitive.

We study categories of similarity spaces in which each space can have its own target domain of possible similarity values. The morphisms are some analogues of non-expanding functions, but modulo some rescaling that is needed to compare the similarity values of different spaces.

Definitions: Let $\mathcal{X} = (X, S_\mathcal{X}, \sigma_\mathcal{X})$ and $\mathcal{Y} = (Y, S_\mathcal{Y}, \sigma_\mathcal{Y})$ be two gss.
A function $f : X \to Y$ is *globally continuous (GC)* w.r.t. $\mathcal{X}$ and $\mathcal{Y}$ if there is a continuous $\varphi : S_\mathcal{X} \to S_\mathcal{Y}$ s.t. $\varphi(\sigma_\mathcal{X}(x, x')) \le \sigma_\mathcal{Y}(fx, fx')$ for all $x, x' \in X$, and $\varphi(\sigma_\mathcal{X}(x, x)) = \sigma_\mathcal{Y}(fx, fx)$ for all $x \in X$.
It is *right-locally continuous (RLC)* if there is a family $(\varphi^{\mathrm{R}}_x)_{x\in X}$ of continuous $\varphi^{\mathrm{R}}_x : S_\mathcal{X} \to S_\mathcal{Y}$ s.t. $\varphi^{\mathrm{R}}_x(\sigma_\mathcal{X}(x, x')) \le \sigma_\mathcal{Y}(fx, fx')$ for all $x, x' \in X$, and $\varphi^{\mathrm{R}}_x(\sigma_\mathcal{X}(x, x)) = \sigma_\mathcal{Y}(fx, fx)$ for all $x \in X$.
It is *left-locally continuous (LLC)* if there is a family $(\varphi^{\mathrm{L}}_x)_{x\in X}$ of continuous $\varphi^{\mathrm{L}}_x : S_\mathcal{X} \to S_\mathcal{Y}$ s.t. $\varphi^{\mathrm{L}}_x(\sigma_\mathcal{X}(x', x)) \le \sigma_\mathcal{Y}(fx', fx)$ for all $x, x' \in X$, and $\varphi^{\mathrm{L}}_x(\sigma_\mathcal{X}(x, x)) = \sigma_\mathcal{Y}(fx, fx)$ for all $x \in X$.
It is *locally continuous (LC)* if it is both RLC and LLC.

A *similarity space* proper is a gss (X, S, σ) where S is a continous lattice with its Scott topology. A generalization of the well-known injectivity of continous lattices can be used to prove various equivalences:

- RLC ⇔ continuous w.r.t. the neighborhood system induced by the right pre-open balls;
- LLC ⇔ continuous w.r.t. the neighborhood system induced by the left pre-open balls;
- LC ⇔ pairwise continuous w.r.t. both neighborhood systems;
- GC ⇔ uniformly continuous in the following sense:
  for every x in X and open v of $S_\mathcal{Y}$ containing $\sigma_\mathcal{Y}(fx, fx)$, there is an open u of $S_\mathcal{X}$ containing $\sigma_\mathcal{X}(x, x)$ with the property $\sigma_\mathcal{X}(x_1, x_2) \in u \Rightarrow \sigma_\mathcal{Y}(fx_1, fx_2) \in v$.
- The category of locally transitive similarity spaces + LC functions is equivalent to BiTop (bitopological spaces + pairwise continuous functions).
- The category of *symmetric* locally transitive similarity spaces + LC functions is equivalent to Top (topological spaces + continuous functions).
- The category of globally transitive similarity spaces with a constant value for $\sigma(x, x)$ + GC functions is equivalent to the category of quasi-uniform spaces + uniformly continuous functions. The equivalence restricts to the symmetric spaces on both sides. (Symmetric quasi-uniform spaces are uniform spaces.)